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Lectures	Tues. 11:30-13:00, kl. Hörsaal A
Excerices	Thurs.14:00-15:30, kl. Hörsaal A
ILIAS	`https://ilias.studium.kit.edu/goto.php?target=crs_1945182&client_id=produktiv`

Sheet 3 – Due 15.12.2022

1) **Age of the Universe**

The Friedmann equations describe a special solution of Einstein's field equations. An important prerequisite is the assumption of the cosmological principle, which determines a homogeneous and isotropic Universe. They allow a temporal development of the universe that depends on the energy content.
In the following we first consider a Universe that allows curvature ($k \neq 0$):

(a) Show from the Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_M + \rho_\Lambda) - \frac{k}{a^2} \tag{1}$$

as well as the definitions

$$\Omega_M = \frac{8\pi G}{3H_0^2}\rho_{M,0}\ ,\quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}\ ,\quad \Omega_K = -\frac{k}{a_0^2 H_0^2} \tag{2}$$

the relationship $\Omega_M + \Omega_\Lambda + \Omega_K = 1$ which was introduced in the lecture. $\rho_{M,0}$ is the matter density at time t_0 and $\Lambda = 8\pi G\rho_\Lambda$. The energy content associated with radiation is assumed to be vanishing and $\Omega_R = 0$.

(b) Derive from the Friedmann equation and the relationship between redshift and scale parameters

$$\frac{a}{a_0} = \frac{1}{1+z} \qquad \text{where} \qquad a_0 = 1. \tag{3}$$

the following equation

$$H^2 = H_0^2\left[\Omega_M(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]. \tag{4}$$

To do this, use the insights you gained in 1 (a).

(c) Suppose we are in an Einstein de-Sitter universe where: $\Omega_\Lambda = \Omega_K = 0$.
Replace in the definition of the Hubble parameter

$$H = \frac{\dot{a}}{a} \tag{5}$$

the scale parameter a with the redshift z and show that

$$H = -\frac{1}{1+z}\frac{dz}{dt}. \tag{6}$$

and calculate dt/dz.

(d) Calculate the age of the Einstein de-Sitter universe ($\Omega_M = 1, \Omega_\Lambda = \Omega_K = 0$) by integrating dt/dz and using the Hubble parameter $H_0 = 70\,\mathrm{km/(s\,Mpc)}$.

2) Black body radiation when space is expanded

In this task, we want to deal with the thermal universe. Recall the material from Lecture 5.

(a) The blackbody spectrum describes the photon density n in the energy interval $\hbar\,\mathrm{d}\omega$ at temperature T

$$n(\omega, T)d\omega = \frac{1}{\pi^2 c^3}\frac{\omega^2 d\omega}{e^{\hbar\omega/k_b T} - 1}. \tag{7}$$

How do ω and T change when space is expanded as a function of the redshift z? How does this change the spectrum?

(b) What is the differential energy spectrum of blackbody radiation? Derive the Stefan-Boltzmann law of energy density by integrating the energy spectrum.
(Hint: $\int_0^\infty \frac{\xi^3 d\xi}{e^\xi - 1} = \frac{\pi^4}{15}$).

(c) Calculate the fraction Ω_γ of the photons of the cosmic microwave background ($T = 2.73\,\mathrm{K}$) contribute to the total energy density of the universe. The critical density is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.1 \cdot 10^4\ \mathrm{h}^2\ \frac{\mathrm{eV}}{\mathrm{cm}^3}\ . \tag{8}$$

(d) Use the Stefan-Boltzmann law to derive a relationship between the temperature T and the time t in the radiation-dominated universe. Can a temperature be assigned to the Big Bang?