Exercise to the Lecture: Intro. to Cosmology
KIT, Wintersemester 2022/2023

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Lectures	Tues. 11:30-13:00, kl. Hörsaal A
Excerices	Thurs.14:00-15:30, kl. Hörsaal A
ILIAS	`https://ilias.studium.kit.edu/goto.php?target=crs_1945182&client_id=produktiv`

Sheet 6 – Due 16.02.2023

1) **Structure formation in the early universe**

Derive the conditions under which a density fluctuation of matter in the early universe collapsed under its own gravity. These are called the Jeans criterion.

(a) First, estimate the time scale on which a gravitational collapse occurred by calculating the infall time of the gas cloud. Assume that the pressure is negligible and calculate the time it takes for a shell of mass m of a spherical gas cloud of total mass M to fall freely from a height r_0 to height $r = 0$. Show that this depends only on the initial density.
(*Hint:* The mass within the spherical shell under consideration remains constant in this model, since it also collapses due to gravity; the resulting integral can be solved as
$\int_{x_0}^{0} \frac{\mathrm{d}x}{\sqrt{\frac{1}{x}-\frac{1}{x_0}}} = -\sqrt{\frac{\pi^2 x_0^3}{4}}$)

(b) To prevent the initial density fluctuations from being compensated by pressure waves, the time of incidence must be small compared to the propagation time of sound waves. Calculate the length at which these two times are equal. Use the model of a monatomic ideal gas to determine the speed of sound.

(c) While a (homogeneous) gas cloud collapses, it heats up. In hydrodynamic equilibrium, the gas pressure balances the gravitational force. Calculate the critical radius as a function of density and temperature at which the thermal energy equals the potential energy. Again, assume a single-atom ideal gas. Compare this with the length scale of the previous part of the task.

(d) The dynamics of matter in the early universe can be described by 3 equations, the continuity equation, the Euler equation for a fluid with influence of gravity, and the Poisson equation. By perturbation theory one obtains for the time evolution of a perturbation $\delta = \frac{\Delta\rho}{\rho}$ of the homogeneous density distribution

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}\delta}{\mathrm{d}t} = \left(4\pi G\rho - \frac{k^2 v_s^2}{a^2}\right)\delta, \tag{1}$$

with the scale factor a, the sound velocity v_s and the wavenumber k, which corresponds to the length scale of a disturbance. For which wavenumber does the term on the right-hand side vanish? What is the form of the solutions to this equation for much larger and smaller wavenumbers? How does the expansion of the universe affect the solution?

2) <u>Influence of neutrinos on structure formation</u>

(a) Determine the escape velocity of a typical galaxy (R = 15 kpc, $M = 1 \cdot 10^{11} M_{\text{sol}}$) and in a typical galaxy cluster (R = 5 Mpc, $M = 1 \cdot 10^{14} M_{\text{sol}}$). Compare these with the mean velocity of neutrinos of the cosmic neutrino background (CνB) with assumed masses of 0.1, 1.0, and 10.0 eV. Assume that the neutrinos are Maxwell distributed and not relativistic. With which of the given masses can the neutrino be trapped in the gravitational potential?

(b) What mass would a galaxy cluster (R = 50 Mpc) have to have to capture a CνB neutrino? Compare this to the typical mass of a galaxy cluster.

(c) Sketch the power spectrum of the matter distribution. How and why does the power spectrum change as the relative fraction of ρ_m increases while $\Omega_{\text{tot}} = 1$ remains? How does the power spectrum change with increasing neutrino mass?

(d) From the matter distribution and the results of the WMAP experiment, the upper limit for the neutrino mass is $m_\nu = 0.23\,\text{eV}$. The neutrino density is $n_\nu = 300\,\text{cm}^{-3}$. What is the maximum contribution Ω_ν of neutrinos to the total energy density of the universe? What is the contribution to the baryonic mass?