Exercise to the Lecture: Intro. to Cosmology
KIT, Wintersemester 2023/2024

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Lectures	Tues. 11:30-13:00, kl. Hörsaal A
Excerices	Thurs.14:00-15:30, kl. Hörsaal A
ILIAS	`https://ilias.studium.kit.edu/goto.php?target=crs_2238560&client_id=produktiv`
Sheet 3, due date	07.12.2022

1) Age of the Universe

The Friedmann equations describe a special solution of Einstein's field equations. The equations can be used to calculate the universe's temporal development, which depends on the energy content. An important prerequisite is the assumption of the cosmological principle, i.e. that the Universe is homogeneous and isotropic. In the following we consider a Universe that allows for curvature ($k \neq 0$):

(a) Using the Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_M + \rho_\Lambda) - \frac{k}{a^2} \,, \tag{1}$$

as well as the definitions

$$\Omega_M = \frac{8\pi G}{3H_0^2}\rho_{M,0} \,, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2} \,, \quad \Omega_K = -\frac{k}{a_0^2 H_0^2} \,, \tag{2}$$

show the following relationship: $\Omega_M + \Omega_\Lambda + \Omega_K = 1$. Here, $\rho_{M,0}$ is the matter density at time t_0 and $\Lambda = 8\pi G\rho_\Lambda$. The energy content associated with radiation is assumed to be vanishing, i.e. $\Omega_R = 0$.

(b) From the Friedmann equation and the relationship between redshift and scale parameters

$$\frac{a}{a_0} = \frac{1}{1+z} \qquad \text{where} \qquad a_0 = 1 \,, \tag{3}$$

derive the following equation

$$H^2 = H_0^2\left[\Omega_M(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right] . \tag{4}$$

To do this, use the insights you gained in (a), and use the conservation of matter in the expanding universe to relate $\rho_{M,0}$ with ρ_M.

(c) Suppose we are in an Einstein de-Sitter universe where: $\Omega_\Lambda = \Omega_K = 0$. In the definition of the Hubble parameter $H = \dot{a}/a$, replace the scale parameter a with the redshift z and show that

$$H = -\frac{1}{1+z}\frac{dz}{dt} \,. \tag{5}$$

Then calculate dt/dz.

(d) Calculate the age of the Einstein de-Sitter universe ($\Omega_M = 1, \Omega_\Lambda = \Omega_K = 0$) by integrating dt/dz and using the Hubble parameter $H_0 = 70\,\text{km}/(\text{s Mpc})$.

2) Blackbody radiation in expanding space

In this exercise, we deal with the thermal universe. Recall the material from lectures 4+5+6.

(a) The blackbody spectrum describes the photon density n in the energy interval $\hbar \mathrm{d}\omega$ at temperature T. It is given by

$$n(\omega, T)d\omega = \frac{1}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\hbar\omega/k_b T} - 1}. \tag{6}$$

How do ω and T change when space is expanded as a function of the redshift z? How does this change the spectrum? (Hint: Use the relation for the thermal energy $E = kT$)

(b) What is the differential energy spectrum of blackbody radiation? Derive the Stefan-Boltzmann law ($\rho \propto T^4$) by integrating the differential energy spectrum to obtain the energy density ρ. (Hint: $\int_0^\infty \frac{\xi^3 d\xi}{e^\xi - 1} = \frac{\pi^4}{15}$).

(c) Calculate the photon fraction $\Omega_\gamma = \rho_\gamma / \rho_c$ of the universe's energy density. Here, ρ_γ is the energy density of the CMB photons ($T = 2.73\,\mathrm{K}$), and the critical density is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.1 \cdot 10^4 \ \mathrm{h}^2 \ \frac{\mathrm{eV}}{\mathrm{cm}^3} \ . \tag{7}$$

(d) Use the Stefan-Boltzmann law to derive a relationship between the temperature T and the time t in the radiation-dominated universe (Hint: Use your knowledge from the lecture to get the time-dependence of the scale factor, or use the first Friedmann equation to calculate it). Can a temperature be assigned to the Big Bang?