Exercise to the Lecture: Intro. to Cosmology
KIT, Wintersemester 2023/2024

Prof. G. Drexlin, A. Lohkov, M. Descher, Y. Biondi

Lectures	Tues. 11:30-13:00, kl. Hörsaal A
Excerices	Thurs.14:00-15:30, kl. Hörsaal A
ILIAS	`https://ilias.studium.kit.edu/goto.php?target=crs_2238560&client_id=produktiv`
Sheet 6, due date	08.02.2024

1) Structure formation in the early universe

Derive the conditions under which a density fluctuation of matter in the early universe collapsed under its own gravity. These are called the Jeans criterion.

(a) First, estimate the time scale on which a gravitational collapse occurred by calculating the infall time of the gas cloud. Assume that the pressure is negligible and calculate the time it takes for a shell of mass m of a spherical gas cloud of total mass M to fall freely from a height r_0 to height $r = 0$ (free-fall time). Show that this depends only on the initial density $\rho_0 = M/(\frac{4}{3}\pi r_0^3)$.

Hints:
The mass within the spherical shell under consideration remains constant in this model, since it also collapses due to gravity. Determine the velocity of the shell and use $dt = dr/v$. The resulting integral can be solved as $\int_{x_0}^{0} \sqrt{1/x - 1/x_0}^{-1} \mathrm{d}x = -\sqrt{\pi^2 x_0^3/4}$

(b) To prevent the initial density fluctuations from being compensated by pressure waves, the infall time must be small compared to the propagation time of sound waves. Calculate the length λ at which these two times are equal. The speed of sound can be determined using the model of a monatomic ideal gas (only 3 translational degrees of freedom) with atomic mass μ.

(c) Now we test a different approach to determine the length scale: While a (homogeneous) gas cloud collapses, it heats up. In hydrodynamic equilibrium, the gas pressure balances the gravitational force. Calculate the critical radius r_c (as a function of density and temperature) at which the internal (thermal) energy equals the gravitaional binding (potential) energy. Again, assume a monatomic ideal gas with molecule mass μ. The result can be compared with the length scale from the previous task.

(d) The dynamics of matter in the early universe can be described by 3 equations, the continuity equation, the Euler equation for a fluid with influence of gravity, and the Poisson equation. By perturbation theory one obtains for the time evolution of a perturbation $\delta = \frac{\Delta\rho}{\rho}$ (density contrast) of the homogeneous density distribution

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}\delta}{\mathrm{d}t} = \left(4\pi G\rho - \frac{k^2 v_s^2}{a^2}\right)\delta, \tag{1}$$

with the scale factor a, the sound velocity v_s and the wavenumber $k = \frac{2\pi a}{\lambda}$, which corresponds to the length scale of a disturbance. (Note: The scale factor increases all lengths and therefore appears as normalization in the wave number.)

- For which wavelength λ_J does the term on the right-hand side vanish?
- What is the general form of the solutions to this equation for much larger and smaller k? For this consider the case that $\dot{a} = 0$.
- How does the expansion of the universe affect the solution?

2) **<u>Influence of neutrinos on structure formation</u>** (bonus exercise)

(a) Determine the escape velocity of a typical galaxy (R = 15 kpc, $M = 1 \times 10^{11} M_{sol}$) and in a typical galaxy cluster (R = 5 Mpc, $M = 1 \times 10^{14} M_{sol}$). Compare these with the mean velocity of neutrinos of the cosmic neutrino background (CνB) with assumed masses of 0.1, 1.0, and 10.0 eV. Assume that the neutrino velocities are non-relativistic and follow a Maxwell-Boltzmann distribution (CνB temperature today: T = 1.95 K). With which of the given masses can the neutrino be trapped in the gravitational potential?

(b) What mass (expressed in solar masses) would a galaxy cluster (R = 50 Mpc) have to have to capture a CνB neutrino? Compare this to the typical mass of a galaxy cluster.

(c) Sketch the power spectrum of the matter distribution. How and why does the power spectrum change as the relative fraction of Ω_m increases while $\Omega_{tot} = 1$ remains? How does the power spectrum change with increasing neutrino mass?

(d) From the matter distribution and the results of the WMAP experiment, the upper limit for the neutrino mass is m_ν = 0.23 eV. The neutrino density is $n_\nu = 300\,\mathrm{cm}^{-3}$. What is the maximum contribution Ω_ν of neutrinos to the total energy density of the universe ($\rho_c = 5390\,\mathrm{eV/cm^3}$)? How large of a contribution is this relative to the overall baryon density?