Problem set 1

Submission deadline: 1 May, 09:45
Discussion of solutions: 4 May, 11:30

Problem 1: Symmetries of the Riemann tensor

a) Using the explicit form of the Christoffel symbol in terms of the metric, show that $\Gamma^\mu_{\nu\rho}$ is symmetric in the two lower indices: $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$.

b) Using the explicit form of $R^\lambda_{\nu\rho\sigma}$ in terms of Christoffel symbols, show that

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \; . \tag{1}$$

Let us define the fully covariant Riemann tensor

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda_{\nu\rho\sigma} \; . \tag{2}$$

Using a so-called *locally inertial frame*, it can be shown that this tensor satisfies the block interchange symmetry $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$.

c) Show that $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$.

d) Conclude that in two dimensions (i.e. for $\mu, \nu, \rho, \sigma \in \{0, 1\}$) the Riemann tensor is fully determined by the component R_{1010}.

e) How many independent components are there in three dimensions?

Problem 2: Examples of curvature

Consider flat three-dimensional space. In cartesian coordinates $(y_1, y_2, y_3) = (x, y, z)$, the metric is simply given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; , \tag{3}$$

for which all Christoffel symbols vanish. Let us instead consider spherical coordinates $(x_1, x_2, x_3) = (r, \theta, \phi)$ with

$$x = r \sin\theta \cos\phi \, , \tag{4}$$
$$y = r \sin\theta \sin\phi \, , \tag{5}$$
$$z = r \cos\theta \; . \tag{6}$$

a) Use the relation $g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu} \mathrm{d}y^\mu \mathrm{d}y^\nu$ to show that the metric in spherical coordinates is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \; . \tag{7}$$

b) Calculate all non-vanishing Christoffel symbols. Show that $R_{\theta\phi\theta\phi} = 0$.

In fact, all components of the Riemann tensor vanish, as expected in flat space.

Let us now instead consider the two-dimensional surface of a sphere with constant radius R, i.e. $(x_1, x_2) = (\theta, \phi)$. By setting $\mathrm{d}r = 0$ in the calculation above we obtain the metric as

$$g_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} . \tag{8}$$

c) Show that in this case $R_{\theta\phi\theta\phi} \neq 0$, representing the fact that the surface of the sphere is not flat.

d) Calculate the resulting Ricci tensor and Ricci scalar.

Problem 3: Geodesics

Consider a particle moving along a path from X^μ_{start} to X^μ_{end}. The path can be parametrised by a function $X^\mu(\lambda)$ with $0 \leq \lambda \leq 1$ such that $X^\mu(0) = X^\mu_{\text{start}}$ and $X^\mu(1) = X^\mu_{\text{end}}$. The length of the path (i.e. the proper time that passes for the particle) is given by

$$\tau = \int_0^1 \mathrm{d}\lambda \sqrt{g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \equiv \int_0^1 \mathrm{d}\lambda L[X^\mu, \dot{X}^\nu] \tag{9}$$

with $\dot{X}^\mu = \mathrm{d}X^\mu/\mathrm{d}\lambda$.

a) Make use of the relation $\mathrm{d}\tau/\mathrm{d}\lambda = L$ to show that

$$g_{\mu\nu}U^\mu U^\nu = 1 , \tag{10}$$

where $U^\mu = \mathrm{d}X^\mu/\mathrm{d}\tau$.

Out of all possible paths, the one that requires the shortest proper time must satisfy the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) - \frac{\partial L}{\partial X^\mu} = 0. . \tag{11}$$

b) Show that the Euler-Lagrange equation expressed in terms of U^μ is just the geodesic equation.