Problem set 3

Submission deadline: 12 June, 9:45
Discussion of solutions: 15 June, 11:30

Problem 7: Chemical potential

a) Show that for a relativistic fermion with $T \gg m$, the equilibrium number density can be written as

$$n = \frac{gT^3}{2\pi^2} \int_{-y}^{\infty} \mathrm{d}q \frac{q^2 + 2yq + y^2}{e^q + 1} , \tag{1}$$

where $y = \mu/T$ and $q = x - y$ with $x = E/T$.

b) Find the corresponding expression for the number density of anti-particles $\bar{n}$ in terms of y and $r = x + y$.

c) Show that

$$n - \bar{n} = \frac{gT^3}{6\pi^2} \left[\pi^2 \left(\frac{\mu}{T} \right) + \left(\frac{\mu}{T} \right)^3 \right] . \tag{2}$$

d) Find the corresponding expression for bosons.

e) Taking $\Omega_b = 0.05$ for the present-day abundance of baryons, assuming that all baryons are protons and that the universe is electrically neutral, calculate the present-day number density of electrons, $n_{e,0}$.

f) Taking $g_{*,s} = 3.94$ and assuming that the present-day number density of positrons $\bar{n}_{e,0}$ is negligible, calculate $(n_{e,0} - \bar{n}_{e,0})/s_0$. Which special property makes this quantity particularly interesting?

g) Calculate $n_e - \bar{n}_e$ at the beginning of Big Bang Nucleosynthesis ($T = 5\,\mathrm{MeV}$).

h) Determine the corresponding chemical potential of electrons.

Hint:

$$\int_0^{\infty} \frac{x}{e^x + 1} \mathrm{d}x = \frac{\pi^2}{12} , \qquad \int_0^{\infty} \frac{x}{e^x - 1} \mathrm{d}x = \frac{\pi^2}{6} . \tag{3}$$

Problem 8: Cannibal domination

Consider a non-relativistic particle species ϕ that does not interact with any other particle species in the universe. Let us consider the effect of number changing processes, such as $3\phi \to 2\phi$.

a) Show that if these processes are in equilibrium, the chemical potential of ϕ must vanish.

b) Neglecting the contribution from pressure, show that entropy conservation implies

$$\sqrt{\frac{T}{m}} e^{-m/T} a^3 = \bar{a}^3 \ , \tag{4}$$

where $\bar{a}$ is a constant.

c) Show that for $m \gg T$ the solution of this equation is approximately

$$\frac{T}{m} \approx \frac{1}{3 \log(a/\bar{a})} \ . \tag{5}$$

d) Show that the energy density of ϕ decreases more rapidly with increasing a than usual non-relativisitc matter, but more slowly than radiation.

A particle species with number-changing processes is sometimes called "cannibalistic": As the universe cools down the particles "eat" each other to "stay warm", i.e. they convert rest mass into kinetic energy.

Problem 9: BBN constraints on ΔN_{eff}

a) The neutron freeze-out temperature T_n is determined from the condition $\Gamma = H$. Determine how T_n depends on the relativistic degrees of freedom g_*.

b) Assuming that the temperature of deuterium burning $T_D = 80\,\text{keV}$ is approximately independent of g_*, determine how the corresponding time t_D depends on g_*.

c) Now write $g_* = 3.38 + \Delta g_*$ to calculate t_D/τ_n and $\Delta m/T_n$ to linear order in Δg_*.

d) Writing $Y_p(g_*) = Y_p(\Delta g_* = 0) + \Delta Y_p$, calculate $\Delta Y_p / Y_p(\Delta g_* = 0)$ to linear order in Δg_*.

e) Use the observational bound $\Delta Y_p / Y_p < 1.2\%$ to derive a bound on Δg_*. Translate this bound into a bound on ΔN_{eff}.

For the case $\Delta g_* = 0$, you can use the following values without calculation: $T_n = 0.75\,\text{MeV}$, $\Delta m = 1.3\,\text{MeV}$, $t_D = 150\,\text{s}$, $\tau_n = 880\,\text{s}$.