Problem set 4

Submission deadline: 26 June, 9:45
Discussion of solutions: 29 June, 11:30

Problem 10: Helium recombination

a) Helium comes in three ionisation states: Neutral He, simply ionized He^{+} and doubly ionized He^{++}. What are the degrees of freedom of each state?

b) Write down the Saha equation for the number density of He^{+} in terms of the number densities of He^{++} and electrons. Why is it a good approximation to assume that n_e is independent of the ionization state of helium?

c) Using numerical methods, find the temperature when the number density of He^{+} first exceeds the number density of He^{++}. Estimate the corresponding age of the universe.

d) Find the temperature below which the majority of helium in the universe is neutral.

The ionization energies of He are $B_1 = 24.5\,\mathrm{eV}$ *for the first ionization and* $B_2 = 54.2\,\mathrm{eV}$ *for the second ionization.*

Problem 11: Horizon problem

a) Calculate the size of the particle horizon $l_{H,\mathrm{dec}}$ (see problem 5) at the moment of photon decoupling. A symbolic answer is sufficient.

b) Consider two points separated by $l_{H,\mathrm{dec}}$ at decoupling. What is their physical distance $d_{H,\mathrm{dec}}$ in the present universe?

c) Calculate the present-day distance of the surface of last scattering d_{dec} (assuming for simplicity that the universe is flat and matter dominated).

d) Estimate the number of regions on the surface of last scattering that were causally disconnected at the time of decoupling.

The fact that causally disconnected parts of the CMB have exactly the same temperature is known as the horizon problem. The theory of inflation offers an elegant solution.

Problem 12: Comoving gauge

In a general gauge the perturbed energy-momentum tensor is given by[1]

$$T^\mu_{\ \nu} = \begin{pmatrix} \bar\rho & 0 \\ 0 & -\bar P \delta^i_j \end{pmatrix} + \begin{pmatrix} \delta\rho & -(\bar\rho + \bar P)(v_i - B_i) \\ (\bar\rho + \bar P)v_i & -\delta P \delta^i_j \end{pmatrix} , \tag{1}$$

where $B_i = \partial_i B$ is a metric perturbation: $\delta g_{0i} = -B_i$.
Consider an infinitesimal coordinate transformation

$$X^\mu \to \tilde X^\mu = X^\mu + \xi^\mu(\eta, \mathbf{x}) \tag{2}$$

with $\xi^0 = T$ and $\xi^i = L^i$. Under this change of coordinates, the energy momentum tensor transforms as

$$T^\mu_{\ \nu}(X) = \frac{\partial X^\mu}{\partial \tilde X^\alpha} \frac{\partial \tilde X^\beta}{\partial X^\nu} \tilde T^\alpha_{\ \beta}(\tilde X) . \tag{3}$$

a) Show that the coordinate transformation leads to the following transformations of matter perturbations:

$$\begin{aligned} \delta\rho &\to \delta\rho - T\bar\rho' \\ \delta P &\to \delta P - T\bar P' \\ v_i &\to v_i - L_i' \end{aligned} \tag{4}$$

b) Show that the combination

$$\Delta \equiv \frac{\delta\rho}{\bar\rho} + \frac{\bar\rho'}{\bar\rho}(B - v) , \tag{5}$$

where $v_i = \partial_i v$, and $B_i = \partial_i B$ is a metric perturbation, is invariant under coordinate transformations.

c) Show that it is possible to choose a gauge, where $v = B = 0$, such that $\Delta = \delta = \frac{\delta\rho}{\bar\rho}$. This gauge is called “comoving gauge”.

[1] For the derivation, see `https://www.mv.helsinki.fi/home/hkurkisu/CosPer.pdf`, section 9.