Problem set 5

Submission deadline: 10 July, 9:45
Discussion of solutions: 13 June, 11:30

Problem 13: Horizon exit

Consider a flat ΛCDM universe with

$$H = H_0 \sqrt{\Omega_{\rm rad} \left(\frac{a_0}{a}\right)^4 + \Omega_{\rm m} \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda} \,. \tag{1}$$

a) Show that the conformal Hubble rate $\mathcal{H} = aH$ has a minimum for $a < a_0$. Find the corresponding value of a for $\Omega_{\rm m} = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_{\rm rad} \approx 0$ and $H_0 = 70\,{\rm km\,s^{-1}\,Mpc^{-1}}$.

b) Argue that this means that there are perturbations that were subhorizon at some time in the past but are now superhorizon.

c) Find the smallest conformal momentum $k_{\rm min}$ of any perturbation that was subhorizon at some point in the past. Calculate the corresponding physical size of this perturbation in the present universe.

d) Compare your result to the physical size of perturbations that exit the horizon today.

Problem 14: Tensor perturbations

Let us consider tensor perturbations of the metric

$$ds^2 = a^2(\eta)\left[{\rm d}\eta^2 - (\delta_{ij} + \hat{E}_{ij}){\rm d}x^i{\rm d}x^j\right] \,, \tag{2}$$

where $\hat{E}_{ij}$ is a traceless and transverse tensor. Any such tensor can be decomposed into a sum of two different polarization states ($\alpha = +, \times$):

$$\hat{E}_{ij}(\eta, \mathbf{x}) = \sum_\alpha h_\alpha(\eta, \mathbf{x}) e^\alpha_{ij} \,. \tag{3}$$

The linearised Einstein tensor is given by $\delta G_{00} = 0$, $\delta G_{0i} = 0$ and

$$\delta G_{ij} = \frac{1}{2a^2}\left(\partial_\eta^2 \hat{E}_{ij} + 2\mathcal{H}\partial_\eta \hat{E}_{ij} - \partial_k\partial_k \hat{E}_{ij}\right) \,. \tag{4}$$

a) Assuming that there are no tensor perturbations in the matter, show that the Einstein equation implies

$$h''_\alpha(\eta, k) + 2\mathcal{H}h'_\alpha(\eta, k) + k^2 h_\alpha(\eta, k) = 0 \,. \tag{5}$$

b) Argue that the third term is negligible in the superhorizon regime, leading to the solution $h_\alpha = h^{(i)}_\alpha$.

c) Using the ansatz $h_\alpha(\eta,k) = f_\alpha(\eta,k)/a(\eta)$, show that an approximate solution in the subhorizon regime is given by

$$h_\alpha(\eta,k) = \frac{A}{a(\eta)}\cos(k\eta) + \frac{B}{a(\eta)}\sin(k\eta)\ . \tag{6}$$

d) Find the constants A and B by requiring that the superhorizon solution is recovered in the limit $\eta \to 0$ (you can assume radiation domination for small η).

Since tensor perturbations of the metric can exist in vacuum (i.e. for vanishing energy-momentum tensor), they are called ***gravitational waves****. Possible evidence for gravitational waves from the early universe was reported last week by the NANOGrav collaboration, see* `https://arxiv.org/pdf/2306.16213.pdf` and `https://arxiv.org/pdf/2306.16219.pdf`.

Problem 15: Evolution of dark matter perturbations

After photon decoupling, the equations of energy and momentum conservation for DM and baryons read

$$\delta_b' - k^2 v_b = 3\Phi' \qquad \delta_{\rm DM}' - k^2 v_{\rm DM} = 3\Phi' \tag{7}$$

$$v_b' + \frac{2}{\eta}v_b = -\Phi \qquad v_{\rm DM}' + \frac{2}{\eta}v_{\rm DM} = -\Phi \tag{8}$$

while the first Einstein equation gives

$$k^2\Phi = -\frac{6}{\eta^2}\left(\frac{\Omega_{\rm DM}}{\Omega_{\rm m}}\delta_{\rm DM} + \frac{\Omega_b}{\Omega_{\rm m}}\delta_b\right)\ . \tag{9}$$

a) Show that for a subhorizon mode these equations imply

$$\delta_{\rm DM}'' + \frac{2}{\eta}\delta_{\rm DM}' - \frac{6}{\eta^2}\delta_{\rm DM} = -\frac{6}{\eta^2}\frac{\Omega_b}{\Omega_{\rm m}}\Delta\ , \tag{10}$$

where $\Delta = \delta_{\rm DM} - \delta_b$.

b) Show that the general solution of this equation is given by

$$\delta_{\rm DM} = \frac{\Omega_b}{\Omega_m}\Delta + \alpha\eta^2 + \frac{\beta}{\eta^3}\ , \tag{11}$$

where α and β are numerical constants.

c) By matching to initial conditions (i.e. evaluating eqs. (7) and (8) at photon decoupling), show that

$$\alpha = \frac{1}{\eta_{\rm dec}^2}\left[\frac{\Omega_{\rm DM}}{\Omega_m}\delta_{\rm DM}^{\rm dec} + \frac{\Omega_b}{5\Omega_m}\left(3\delta_b^{\rm dec} + k^2\eta_{\rm dec}v_b^{\rm dec}\right)\right] \tag{12}$$

Hint: You can use the fact that dark matter perturbations are proportional to a during matter domination, such that

$$\left(\delta_{\rm DM}'\right)^{\rm dec} = \left(\frac{a'}{a}\delta_{\rm DM}\right)^{\rm dec} = \frac{2}{\eta_{\rm dec}}\delta_{\rm DM}^{\rm dec}\ . \tag{13}$$