Problem set 6

Submission deadline: 24 July, 9:45
Discussion of solutions: To be confirmed

Problem 16: Perturbed photon geodesics

Let $P^\mu = (P^0, P^i)$ denote the four-momentum of a photon. We define the three-momentum p via $p^2 = -g_{ij}P^iP^j$ and the requirement that p^i and P^i point in the same direction (given by the unit vector $\hat{p}^i$).

a) Show that for a perturbed metric in Newtonian gauge, the condition $P^\mu P_\mu = 0$ implies

$$P^0 = \frac{p}{a}(1-\Psi) \tag{1}$$

$$P^i = \frac{p\hat{p}^i}{a}(1+\Phi)\ . \tag{2}$$

b) Using the Christoffel symbols

$$\Gamma^0_{00} = \mathcal{H} + \frac{\partial\Psi}{\partial\eta} \tag{3}$$

$$\Gamma^0_{i0} = \frac{\partial\Psi}{\partial x^i} \tag{4}$$

$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \left[\frac{\partial\Phi}{\partial\eta} + 2\mathcal{H}(\Phi+\Psi)\right]\delta_{ij} \tag{5}$$

show that

$$\Gamma^0_{\alpha\beta}P^\alpha P^\beta = -\frac{p^2}{a^2(1+2\Psi)}\left[-2\mathcal{H} + \frac{\partial\Phi}{\partial\eta} - \frac{\partial\Psi}{\partial\eta} - 2\hat{p}^i\frac{\partial\Psi}{\partial x^i}\right]\ . \tag{6}$$

c) Use this result to show that the geodesic equation

$$P^0\frac{\mathrm{d}P^0}{\mathrm{d}\eta} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta \tag{7}$$

becomes

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}\eta} = -\mathcal{H} - \hat{p}^i\frac{\partial\Psi}{\partial x^i} + \frac{\partial\Phi}{\partial\eta}\ . \tag{8}$$

Problem 17: Optical depth

After recombination the universe is neutral (and hence transparent to photons) until structure formation leads to reionization. As a result, CMB photons can scatter off free electrons, which changes their direction. The resulting mixing of photons from different points on the last-scattering surface reduces CMB anisotropies.

To estimate the magnitude of this effect, we can assume that reionization happens suddenly at $z = z_{\rm reio}$, and that all hydrogen atoms are ionized, while all helium atoms remain neutral:

$$n_e(z) = \begin{cases} n_H & z < z_{\rm reio} \\ 0 & z \geq z_{\rm reio} \end{cases} \tag{9}$$

The probability for a CMB photon to scatter is then given by $\exp(-\tau_{\rm reio})$, where

$$\tau_{\rm reio} = \sigma_{\rm T} \int_{t_{\rm dec}}^{t_0} {\rm d}t n_e(t) = \sigma_{\rm T} \int_0^{z_{\rm dec}} n_e(z) \frac{1}{(1+z)H(z)} {\rm d}z \tag{10}$$

with the Thomson cross section $\sigma_{\rm T} = 6.65 \times 10^{-25}\,{\rm cm}^2 = 1713\,{\rm GeV}^{-2}$.

a) Calculate $\tau_{\rm reio}$ for $z_{\rm reio} = 10$ using the approximate ΛCDM parameters $\Omega_m = 0.3$, $\Omega_b = 0.05$, $\Omega_\Lambda = 0.7$, $H_0 = 70\,{\rm km\,s^{-1}\,Mpc^{-1}} = 1.4 \times 10^{-42}\,{\rm GeV}$.

   Hint: You can express n_H in terms of n_γ using the information extracted from Big Bang Nucleosynthesis.

b) Reionization cannot mix photons from regions of space that are not causally connected at reionization. Argue that this means that CMB multipoles $\ell < \ell_{\rm reio} = \pi\eta_0/\eta_{\rm reio}$ are not affected by reionization.

c) Calculate $\ell_{\rm reio}$.

The parameter z_{reio} constitutes the sixth (and final) independent parameter of the ΛCDM model.

Problem 18: Bounds on the dark matter mass

If dark matter particles participate in the weak interactions, their annihilation cross section into Standard Model fermions is given by

$$\langle \sigma v \rangle = \frac{g^4\, m_{\rm DM}^2}{16\pi\, \cos^4\theta_{\rm W}\, M_Z^4} \times C\,, \tag{11}$$

where $M_Z = 91\,\mathrm{GeV}$ is the Z-boson mass, $g \approx 0.65$ is the weak gauge coupling, $\theta_{\rm W} \approx 0.50$ is the Weinberg angle and C is a numerical constant that depends on the number of fermions that are kinematically accessible. For dark matter masses of the order of a few GeV one finds $C \sim 6$.

a) Estimate the relic abundance $\Omega_{\rm DM} h^2$ from thermal freeze-out.

b) To be consistent with observations, $\Omega_{\rm DM} h^2$ must not exceed the total amount of DM in the Universe, i.e. $\Omega_{\rm DM} h^2 \leq 0.12$. Use this requirement to obtain the so-called *Lee-Weinberg bound* on $m_{\rm DM}$.

It can be shown in a very general way from the requirement of unitarity (i.e. the conservation of probability) that the annihilation cross section for a Majorana fermion must satisfy the inequality

$$\sigma < \frac{\pi}{m_{\rm DM}^2\, v_{\rm rel}^2}\,, \tag{12}$$

where $v_{\rm rel} = |\mathbf{v}_{\rm rel}|$ is the relative velocity between two dark matter particles, which follows the distribution

$$f(\mathbf{v}_{\rm rel}) = \left(\frac{x}{4\pi}\right)^{3/2} \exp\left(-\frac{x\, v_{\rm rel}^2}{4}\right) \tag{13}$$

with $x = m_{\rm DM}/T$.

c) Derive an upper bound on $\langle \sigma v_{\rm rel} \rangle$ as a function of $m_{\rm DM}$.

d) Calculate a lower bound on the thermal abundance of a dark matter particle with mass $m_{\rm DM}$. Combine this result with the measured value $\Omega_{\rm DM} h^2 = 0.12$ to derive the so-called *unitarity bound* on the WIMP mass.