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Lecture Notes on Cosmology

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Contents

Preface

The following notes are based on the lecture course Cosmology
These notes are still under development and will continuously be improved. If anything is unclear, or if you spot a typo, please send me an email to felix.kahlhoefer@kit.edu.

1 Introduction

1.1 Units and Conventions

Will use natural units : $c = \hbar = k_B = 1$

$$\Rightarrow [\text{mass}] = [\text{momentum}] = [\text{temperature}] = [\text{energy}] = \text{GeV}$$

Conversion: $1\ \text{GeV} = 1.8 \cdot 10^{-24}\ \text{g} = 1.2 \cdot 10^{13}\ \text{K}$

$$[\text{time}] = [\text{distance}] = \left[\text{energy}^{-1}\right] = \text{GeV}^{-1}$$

Conversion: $1\ \text{GeV}^{-1} = 2.0 \cdot 10^{-14}\ \text{cm} = 6.6 \cdot 10^{-25}\ \text{s}$

$\Rightarrow$ Newton's constant of gravity:

$$G = 6.67 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} = 6.71 \cdot 10^{-39}\ \text{GeV}^{-2}$$

Convenient to write $G = M_{\text{pl}}^{-2}$ with

$$\boxed{M_{\text{pl}} = 1.22 \cdot 10^{19}\ \text{GeV} \quad (\text{Planck mass})}$$

Will sometimes need astrophysical units

$$1\ \text{pc} = 3.1 \cdot 10^{18}\ \text{cm} \qquad 1\ M_\odot = 1.99 \cdot 10^{30}\ \text{kg}$$

1.2 The present universe

Observations:

- At sufficiently large scales, universe is homogenous (same everywhere) & isotropic (same in every direction)
- The universe expands

  ↳ galaxies "move away" from us

$$\Rightarrow \text{Doppler effect: } \underbrace{\lambda_{\text{ab}}}_{\text{absorption}} > \underbrace{\lambda_{\text{em}}}_{\text{emission}}$$

$$\text{Define redshift} \quad z = \frac{\lambda_{\text{ab}}}{\lambda_{\text{em}}} - 1$$

  ↳ Hubble's law:

$$z \overset{z \ll 1}{=} H_0 \cdot r$$

$$\text{with Hubble constant } H_0 \approx 67 \pm 1\ \frac{\text{km}}{\text{s} \cdot \text{Mpc}}$$

Convenient to define

$$h = \frac{H_0}{100\ \frac{\text{km}}{\text{s}\cdot\text{Mpc}}} = 0.67 \pm 0.01 \qquad \Rightarrow h^2 \approx 0.5$$

Note: $[H_0] = [\text{rate}] = \left[\text{time}^{-1}\right]$

$\Rightarrow H_0^{-1} \approx 1.4 \cdot 10^{10}$ yrs defines typical time scale (age of the universe)

1.3 Content of the universe

Universe filled with photons following (almost perfect) blackbody spectrum of temperature $T_0 = 2.7255 \pm 0.0006$ K

↳ Cosmic Microwave Background (CMB)

- ⇒ Confirms isotropy of universe at $< 10^{-4}$
- ⇒ Contains huge wealth of information about early universe

Expect also Cosmic Neutrino Background (not yet detected).

Dominant contribution to total energy budget:

- Visible matter: 5%
  - Baryons (i.e. nuclei) but no anti-baryons
  - Electrons ensures charge neutrality
  - Dominant form: Diffuse gas of H and He
  - Heavier elements very rare
- Dark matter: 25%
  - Accounts for "*missing mass*" needed to stabilise galaxies and galaxy clusters
  - Must be non-baryonic, non-relativistic and very weakly interacting
  - Unknown elementary particle?
- Dark energy: 70%
  - Uniformly fills space ("*vacuum energy*")
  - Accounts for (accelerated) expansion
  - Fundamental theory completely unknown

1.4 Universe in the past

Early universe was denser and hotter.

For $T \gtrsim 1$ eV : No bound atoms $\to$ free electrons & nuclei

$T \gtrsim 100$ keV : No bound nuclei $\to$ free protons & neutrons

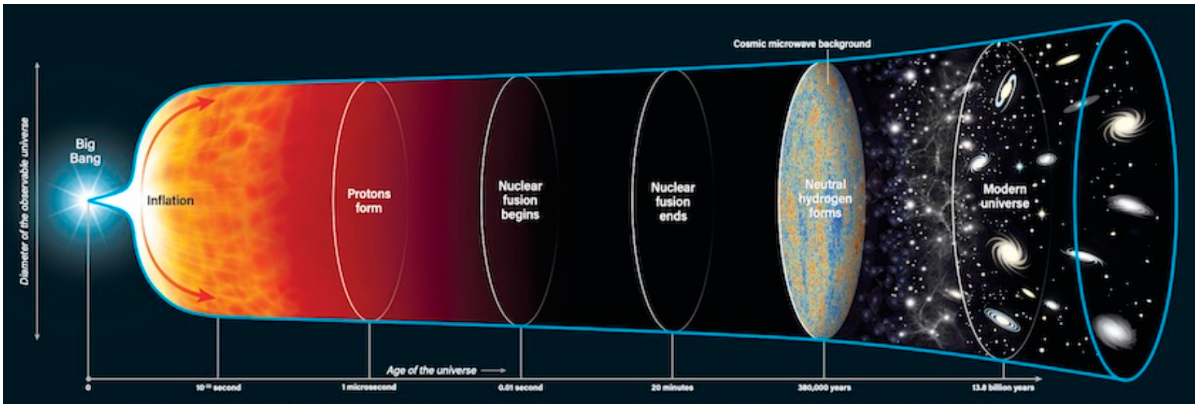
$T \gtrsim 100$ MeV : No bound baryons $\to$ free quarks & gluons

Even higher temperatures: Speculative

↳ Electroweak phase transition ?

↳ Dark matter production ?

↳ Generation of baryon-antibaryon asymmetry ?



Inflation: Sets initial conditions for evolution

2 Brief Introduction to General Relativity

Recap: Special relativity

$$
\begin{aligned}
\text{Define } \mathrm{d}s^2 &= c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \\
&= \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \qquad \text{(summation convention)}
\end{aligned}
$$

$$
\text{with } \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \text{and } \mu,\nu = 0,1,2,3
$$

$\Rightarrow \mathrm{d}s^2$ is invariant under Lorentz transformation

$$
y^\mu = \Lambda^\mu_\nu x^\nu \Rightarrow \mathrm{d}y^\mu = \frac{\partial y^\mu}{\partial x^\nu} x^\nu = \Lambda^\mu_\nu x^\nu, \text{ where } \Lambda^\mu_\rho \Lambda^\nu_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}
$$

Quantities that transform like $\mathrm{d}x^\mu$ are called contravariant vectors

Example: Consider world line $X^\mu(\tau)$ of a particle

$$
\Rightarrow U^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \quad \text{is contravariant vector}
$$

Covariant vectors transform in the opposite way:

$$
\mathrm{d}y_\mu \equiv \eta_{\mu\nu}\,\mathrm{d}y^\mu = \frac{\partial x^\nu}{\partial y^\mu}\,\mathrm{d}x_\nu = \left(\Lambda^{-1}\right)^\nu_\mu \mathrm{d}x_\nu
$$

Generalization to tensors:

$$
T^\mu_\nu(y) = \frac{\partial y^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial y^\nu} T^\rho_\sigma(x) = \Lambda^\mu_\rho \left(\Lambda^{-1}\right)^\sigma_\nu T^\rho_\sigma(x)
$$

$\Rightarrow \eta_{\mu\nu}$ is a rank-2 covariant tensor.
Convenient to define inverse metric $\eta^{\mu\nu}$:

$$
\eta^{\mu\nu}\eta_{\mu\rho} = \delta^\mu_\rho = \text{diag}\ (1,1,1,1)
$$

↳ Can be used to “pull” indices up and down:

$$
T^{\mu\nu} = \eta^{\mu\rho} T^\nu_\rho; \qquad T_{\mu\nu} = \eta_{\mu\rho} T^\rho_\nu
$$

2.1 Non-inertial reference frames

Lorentz transformation do not introduce fictitious forces. Consider instead general transformation $y^\mu = y^\mu(x^\nu)$

$$\Rightarrow \mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}y^\mu\,\mathrm{d}y^\nu = \left(\eta_{\mu\nu}\frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma}\right)\cdot \mathrm{d}x^\rho\,\mathrm{d}x^\sigma$$

$$\equiv \underbrace{g_{\rho\sigma}}_{\text{may depend on x}} \cdot\, \mathrm{d}x^\rho\,\mathrm{d}x^\sigma$$

Consider motion of inertial particle $y^\mu(\tau)$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = 0 \quad \text{(no acceleration)}$$

New coordinate system:

$$\frac{\mathrm{d}^2 y^\mu}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}y^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial y^\mu}{\partial x^\nu}\underbrace{\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}_{=U^\nu}\right)$$

$$= U^\nu\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial y^\mu}{\partial x^\nu}\right) + \frac{\partial y^\mu}{\partial x^\nu}\frac{\mathrm{d}U^\nu}{\mathrm{d}\tau} = 0$$

$$\text{Using}\ \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial y^\mu}{\partial x^\nu} = U^\rho\frac{\partial^2 y^\mu}{\partial x^\rho \partial x^\nu}\ \text{and}\ \frac{\partial y^\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}y^\rho} = \delta^\mu_\rho$$

$$\Rightarrow \frac{\mathrm{d}U^\rho}{\mathrm{d}\tau} + \underbrace{U^\mu U^\nu \underbrace{\left(\frac{\partial^2 y^\sigma}{\partial x^\mu \partial x^\nu}\frac{\partial x^\rho}{\partial y^\sigma}\right)}_{\equiv \Gamma^\rho_{\mu\gamma}}}_{\substack{\neq 0 \text{ in general} \\ \to \text{ non-inertial frame}}} = 0$$

$$\text{Using}\ \frac{\partial g_{\rho\sigma}}{\partial x^\lambda} = \eta_{\mu\nu}\frac{\partial}{\partial x^\lambda}\left(\frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma}\right)$$

$$= \eta_{\mu\nu}\left(\frac{\partial^2 y^\mu}{\partial x^\lambda \partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma} + \frac{\partial^2 y^\nu}{\partial x^\lambda \partial x^\sigma}\frac{\partial y^\mu}{\partial x^\rho}\right) \qquad ;\ \eta_{\mu\nu}\frac{\partial y^\nu}{\partial x^\sigma} = g_{\kappa\sigma}\frac{\partial x^\kappa}{\partial y^\mu}$$

$$= g_{\kappa\sigma}\Gamma^\kappa_{\lambda\rho} + g_{\kappa\rho}\Gamma^\kappa_{\lambda\sigma}$$

we obtain $\boxed{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right) \qquad \text{“Christoffel symbols”}}$

Convenient to define covariant derivative:

Scalars: $\nabla_\mu X = \partial_\mu X$

Vector: $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^{\nu}_{\mu\rho} X^\rho$

$\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^{\rho}_{\mu\nu} X_\rho$

$\Rightarrow \nabla_\mu (X^\nu X_\nu) = \partial_\mu (X^\nu X_\nu)$

Tensor: $\nabla_\mu T^{\nu}_{\rho} = \partial_\mu T^{\nu}_{\rho} + \Gamma^{\nu}_{\mu\sigma} T^{\sigma}_{\rho} - \Gamma^{\sigma}_{\mu\rho} T^{\nu}_{\sigma}$ $\qquad \Rightarrow \nabla_\mu g_{\rho\sigma} = 0$

$$\begin{aligned}
\Rightarrow \frac{\mathrm{d}U^\rho}{\mathrm{d}\tau} + \Gamma^{\rho}_{\mu\nu} U^\mu U^\nu &= \frac{\partial U^\rho}{\partial x^\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + \Gamma^{\rho}_{\mu\nu} U^\mu U^\nu \\
&= U^\mu \left(\underline{\partial_\mu U^\rho} + \Gamma^{\rho}_{\mu\nu} U^\nu \right) \\
&= U^\mu \nabla_\mu U^\rho = 0 \qquad \text{“geodesic equation”}
\end{aligned}$$

Note: Can also write geodesic eq. in terms of
$P^\mu = m U^\mu \Rightarrow P^\mu \nabla_\mu P_\rho = 0 \longrightarrow$ valid also for massless particles

2.2 Curved spacetime

Metric $g_{\mu\nu}$ can describe not only non-inertial frames but also general curved spacetime.

In such a spacetime, covariant derivatives do not commute:

$$\nabla_\mu \nabla_\nu A^\sigma - \nabla_\nu \nabla_\mu A^\sigma = R^{\sigma}_{\mu\nu\rho} A^\rho$$

with $\boxed{R^{\sigma}_{\mu\nu\rho} = \partial_\nu \Gamma^{\sigma}_{\mu\beta} - \partial_\rho \Gamma^{\sigma}_{\mu\nu} + \underline{\Gamma^{\sigma}_{\lambda\nu} \Gamma^{\lambda}_{\mu\rho}} - \Gamma^{\sigma}_{\lambda\rho} \Gamma^{\lambda}_{\mu\nu} \qquad \text{“Riemann tensor”}}$

Interpretation: Consider two particles with separation B^μ traveling with the same velocity U^μ

$$\frac{\mathrm{D}^2 B^\mu}{\mathrm{D}\tau^2} = -R^{\mu}_{\nu\rho\sigma} U^\nu U^\sigma B^\rho$$

$\Lsh \frac{\mathrm{D}}{\mathrm{D}\tau} = U^\nu \nabla_\nu \neq 0$ in curved spacetime

Convenient to define

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} \qquad \text{“Ricci tensor”}$$

$$R = g^{\mu\nu} R_{\mu\nu} \qquad \text{“Ricci scalar”}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad \text{“Einstein tensor”}$$

Comment: Can show that $\nabla^{\mu} G_{\mu\nu} = 0$

2.3 Equivalence principle

Gravity is locally indistinguishable from acceleration (i.e. coordinate transformation to non-inertial frame)

$\Rightarrow$ Effect of gravity fully captured by metric $g_{\mu\nu}$

How does metric depend on gravitating matter?

Consider a perfect fluid with density ρ and pressure p assumed to be homogeneous and isotropic in its rest frame ($U^{\mu} = (1, 0, 0, 0)$)

Define energy-momentum tensor

$$\boxed{T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)}$$

in rest frame. In general frame with velocity U^{μ},

$$T_{\mu\nu} = (\rho + p) U_{\mu} U_{\nu} - p g_{\mu\nu} = \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{momentum density} & \text{stress tensor} \end{array} \right)$$

$E - p$ conservation imply $\nabla^{\mu} T_{\mu\nu} = 0$ for all ν.

Both $G_{\mu\nu}$ and $T_{\mu\nu}$ are covariantly conserved.

Tempting to write $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$ where κ is unknown.

To determine κ^2 consider metric

$$\mathrm{d}s^2 = c^2 \, \mathrm{d}t^2 \left(1 + \frac{2\Phi(\vec{x})}{c^2} \right) - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \qquad \left(\frac{\Phi}{c^2} \ll 1 \right)$$

Find

$$\Gamma^i_{00} = \frac{1}{2} \underbrace{g^{i\sigma}}_{=-\delta^{i\sigma}} \left(\underbrace{(\frac{\partial g_{0\sigma}}{\partial x^0} + \frac{\partial g_{0\sigma}}{\partial x^0}}_{=0 \text{ for } \sigma=i} - \frac{\partial g_{00}}{\partial x^\sigma} \right)$$

$$= \frac{1}{2} \frac{\partial}{\partial x^i} g_{00} = \frac{1}{c^2} \frac{\partial \Phi}{\partial x^i}$$

$\Rightarrow$ Geodesic eq. for non-relativistic particle

$$\ddot{x}^i = -\Gamma^i_{00} U^0 U^0 = -\frac{\partial \Phi}{\partial x^i}$$

↳ Φ acts like Newtonian potential

Now calculate

$$g^{\mu\nu} G_{\mu\nu} = -R = 2\nabla^2 \Phi$$

$$g^{\mu\nu} T_{\mu\nu} = \rho - 3p \overset{\text{non-rel}}{\approx} \rho \qquad \Rightarrow 2\nabla^2 \Phi = \kappa^2 \rho$$

Compare to Poisson eq. $\nabla^2 \Phi = 4\pi G \rho$ where G is Newton's constant.

$$\Rightarrow \kappa^2 = 8\pi G$$

$$\Rightarrow \boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}} \qquad \text{"Einstein equation"}$$

3 The FLRW metric

Recap: Geometry of space-time described by metric $g_{\mu\nu}$ (10 independent functions of $(t, \vec{x})$)

Important simplification: Universe observed to be homogenous & isotropic

↳ $g_{\mu\nu}$ independent of $\vec{x}$

↳ $g_{\mu\nu}$ invariant under rotation

Example: Static flat space

$$\begin{aligned} ds^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\ &= dt^2 - dr^2 - r^2 \underbrace{\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}_{=d\Omega^2} \end{aligned}$$

More general: Allow expansion/contraction of spacial part with time

$$ds^2 = dt^2 - a(t)^2 \left[dr^2 + r^2 d\Omega^2\right] \quad \text{where } a(t) \text{ is the "scale factor"}$$

$a(t)$ is dimensionless $\Rightarrow$ only ratio $\dfrac{a(t_1)}{a(t_2)}$ meaningful.

$$\text{Define } \boxed{H(t) = \frac{\dot{a}(t)}{a(t)}} \qquad (\text{"Hubble rate"})$$

Most general : Allow constant spatial curvature

$$\boxed{ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right] \quad \text{with } k = \begin{cases} +1 & \text{pos. curvature} & \text{(3-sphere)} \\ 0 & \text{flat} & \text{(3-plane)} \\ -1 & \text{neg. curvature} & \text{(3-hyperboloid)} \end{cases}}$$

"Friedmann-Lemaître-Robertson-Walker (FLRW) metric"

Notes:

- For $k \neq 0$, r must be dimensionless and $a(t)$ has dimension of length.
  ↳ Interpretation: $a(t)$ = radius of curvature
- Sometimes convenient to define

$$d\chi \equiv \frac{dr}{\sqrt{1 - kr^2}} \qquad\qquad d\eta = \frac{dt}{a(t)}$$

↳ "coordinate distance" ↳ "conformal time"

$$\Rightarrow ds^2 = a(\eta)^2 \left[d\eta^2 - (d\chi^2 + S_k^2(\chi) d\Omega^2)\right] \qquad \text{with } S_k(\chi) = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$$

Consider particle at rest: $x^\mu = (\tau, x_0, y_0, z_0), u^\mu = (1,0,0,0)$

Geodesic equation:

$$0 = \underbrace{\frac{\mathrm{d}u^\rho}{\mathrm{d}\tau}}_{=0} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = \Gamma^\rho_{00} = \frac{1}{2} g^{\rho\sigma} \left(\partial_0 \underbrace{g_{0\sigma}}_{=\delta_{0\sigma}} + \partial_0 g_{\sigma 0} - \partial_\sigma g_{00} \right) = 0$$

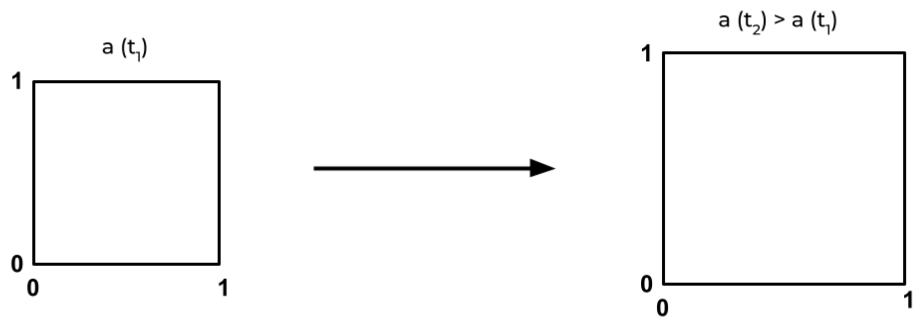
$\Rightarrow$ Particles at rest are free (no forces)

<u>But:</u> Physical distance to origin changes with time t:

$$ds^2 = a(t)^2 \frac{dr^2}{1-kr^2}$$

$$\Rightarrow \underbrace{d(r,t)}_{\substack{\text{physical}\\ \text{distance}}} = a(t) \int_0^r \frac{dr}{\sqrt{1-kr^2}} = a(t) \times \begin{cases} \arcsin r & k=1 \\ r & k=0 \\ \operatorname{arcsinh} r & k=-1 \end{cases}$$

↳ coordinate distance



$$x^i \quad : \text{ "comoving coordinate"}$$

$$X^i = a(t)\, x^i : \text{ "physical coordinate"}$$

$$V^i = \frac{\mathrm{d}X^i}{\mathrm{d}t} = \underbrace{a(t)\frac{\mathrm{d}x^i}{\mathrm{d}t}}_{\text{peculiar velocity}} + \underbrace{H\, X^i}_{\text{"Hubble flow"}}$$

Now consider particle with momentum P^μ

$$0 = P^\alpha \partial_\alpha P^\mu + \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta$$

For $\mu = 0$

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} \underbrace{g^{o\lambda}}_{=\delta^{0\lambda}} \left(\partial_\alpha \underbrace{g_{\beta\lambda}}_{=\delta^{\beta 0}} + \partial_\beta \underbrace{g_{\alpha\lambda}}_{=\delta_{\alpha 0}} - \partial_\lambda g_{\alpha\beta} \right) = -\frac{1}{2} \partial_0 g_{\alpha\beta}$$

$$\Rightarrow \Gamma^0_{00} = \Gamma^0_{0i} = 0\,, \quad \Gamma^0_{ij} = -\frac{1}{2}\partial_0 g_{ij} = \frac{1}{2}\partial_t a(t)^2\, \gamma_{ij} = \dot{a}(t) \cdot a(t) \cdot \underbrace{\gamma_{ij}}_{\substack{\text{spatial} \\ \text{metric}}}$$

$$\Rightarrow 0 = P^\alpha \partial_\alpha P^0 + \dot{a} a \gamma_{ij} P^i P^j$$

Homogeneity of space : $\partial_i P^0 = 0$

Use $P^0 = E, \quad -g_{ij} P^i P^j = a^2 \gamma_{ij} P^i P^j = p^2$ where p is physical 3-momentum.

$$\Rightarrow \quad E\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\dot{a}}{a} p^2 = -H\, p^2$$

$$E^2 - p^2 = m^2 \Rightarrow E\, dE = p\, dp \Rightarrow \frac{\dot{p}}{p} = -\frac{\dot{a}}{a}$$

For $m = 0$: $\quad p = E \sim \dfrac{1}{a}$

↳ Energy of massless particles decreases with increasing scale factor.

For $m \neq 0$: $\quad P^i = mU^i = m\dfrac{\mathrm{d}X^i}{\mathrm{d}\tau} = m\dfrac{\mathrm{d}t}{\mathrm{d}\tau} v^i = \dfrac{mv^i}{\sqrt{1-v^2}} \Rightarrow \dfrac{mv}{\sqrt{1-v^2}} \sim \dfrac{1}{a}$

↳ Peculiar velocity decreases

↳ Particle converges onto Hubble flow

3.1 Redshift

Photons have $\lambda = \dfrac{h}{p} \sim a(t)$

Classical Interpretation: Expansion of space stretches wavelength

Consider photon emitted at time t_i with λ_i

Present universe: $\lambda_0 = \lambda_i \dfrac{a_0}{a(t_i)} = \lambda_i\,(1 + z(t_i))$ with $\boxed{z(t) = \dfrac{a_0}{a(t)} - 1}$ "redshift"

If λ_i is known (e.g. spectral line), we can infer $z(t_i)$ from λ_0

$\Rightarrow$ Infer time since emission

$\Rightarrow$ Infer distance of source

Useful relations: $dz = -\dfrac{a_0}{a^2}\,da = -\dfrac{a_0}{a}\,H\,dt = -(1+z)\,H\,dt$

For nearby sources

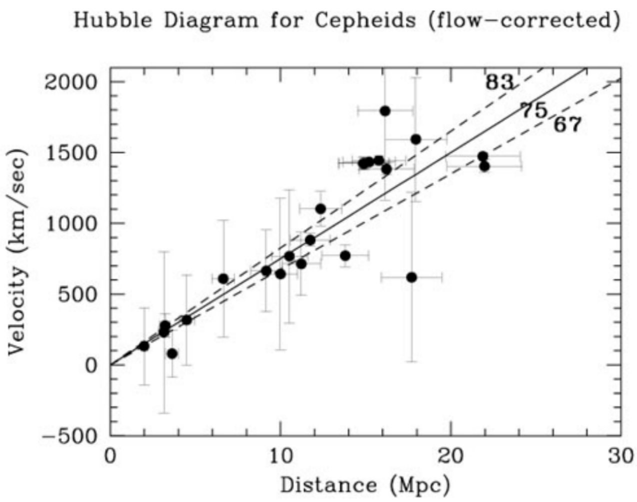
$$a(t_i) = a_0(1 + (t_i - t_0)\cdot H_0 + \dots)\,, \qquad H_0 = \left.\frac{\dot a}{a}\right|_{t=t_0} \quad (\text{"Hubble constant"})$$

$$\Rightarrow z(t_1) \approx H_0 \cdot \underbrace{(t_0 - t_i)}_{\substack{\approx d \\ \text{(distance to} \\ \text{emitter)}}}$$

$\Rightarrow \boxed{z \overset{z\ll 1}{\approx} H_0\,d}$ "Hubble's law"

↳ redshift proportional to distance

↳ can be used to measure H_0 (inaccurate)



Note: $H = \dfrac{\dot a}{a} \Rightarrow [H_0] = [\text{time}]^{-1}\,,\quad [d] = [\text{distance}] \;\Rightarrow [z] = [\text{velocity}]$

Convention: $[H_0] = \text{km}\,\text{s}^{-1}\text{Mpc}^{-1},\quad [d] = \text{Mpc} \quad \Rightarrow [z] = \text{km}\,\text{s}^{-1}$

4 Dynamics of Cosmological Expansion

Recap:
$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$
$$\Rightarrow g_{00} = 1, \quad g_{ij} = -a(t)^2\,\gamma_{ij}$$

What determines $a(t)$?

Einstein equation: $G_{\mu\nu} = 8\pi\, G\, T_{\mu\nu}$

Let's calculate $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

$$\Gamma^i_{0j} = \frac{1}{2} g^{i\mu}\left(\partial_0 g_{\mu j} + \partial_j g_{0\mu} - \partial_\mu g_{0i}\right) \qquad \begin{array}{ll} g^{i\mu} = 0 & \text{for } \mu = 0 \\ g_{0\mu} = 0 & \text{for } \mu \neq 0 \\ g_{0i} = 0 & \end{array}$$
$$= \frac{1}{2} a(t)^{-2} \gamma^{i\mu} \partial_0 \left(a(t)^2 \gamma_{\mu j}\right)$$
$$= \gamma^{i\mu}\gamma_{\mu_j} \frac{1}{2} a(t)^{-2} \cdot 2\, a(t)\dot{a}(t)$$
$$= \delta^i_j \frac{\dot{a}}{a}$$

$$R_{00} = \partial_\lambda \Gamma^\lambda_{00} - \partial_0 \Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{00}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{0\sigma}\Gamma^\sigma_{\lambda 0} \qquad \Gamma^\lambda_{0\lambda} = 0 \text{ for } \lambda = 0, \quad \Gamma^\lambda_{00} = 0$$
$$= -\partial_0 \delta^i_i \frac{\dot{a}}{a} - \delta^i_j \frac{\dot{a}}{a}\, \delta^j_i \frac{\dot{a}}{a}$$
$$= -3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}$$

Analogous calculations: $R_{0i} = 0, \quad R_{ij} = \left(\ddot{a}a + 2\dot{a}^2 + 2k\right)\gamma_{ij}$

$$\Rightarrow R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{ij} R_{ij}$$
$$= -3\frac{\ddot{a}}{a} - a^{-2} \underbrace{\gamma^{ij}\gamma_{ij}}_{=\delta^i_i = 3} \left(\ddot{a}a + 2\dot{a}^2 + 2k\right)$$
$$= -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$
$$\Rightarrow G_{00} = R_{00} - \frac{1}{2} g_{00} R = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

What about $T_{\mu\nu}$?

Consider universe filled with homogeneous fluid with energy density $\rho(t)$ and pressure

$p(t)$ (good approximation on large scales)

Consider "cosmic rest frame": Centre-of mass of fluid at rest $\Rightarrow U^\mu = (1, 0, 0, 0)$

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - g_{\mu\nu}p = \underbrace{\begin{pmatrix} \rho & & & \\ & a^2 p & & \\ & & a^2 p & \\ & & & a^2 p \end{pmatrix}}_{\text{See note at the end of next section}}$$

$\Rightarrow$ For $\mu = \nu = 0$, Einstein equation becomes

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\,\rho - \frac{k}{a^2}} \qquad \text{(Friedmann equation)}$$

↳ Relates two unknown functions of t : $a(t), \rho(t)$

Need additional equation to determine $\rho(t)$

$E - p$ conservation: $\nabla_\mu T^{\mu\nu} = 0$

$$\Rightarrow \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma} = 0$$

Consider $\nu = 0$

$$0 = \partial_0 T^{00} + \Gamma^\mu_{\mu 0} T^{00} + \underbrace{\Gamma^0_{00}}_{=0} T^{00} + \underbrace{\Gamma^0_{0j}}_{=0} T^{0j} + \underbrace{\Gamma^0_{i0}}_{=0} T^{i0} + \Gamma^0_{ij} \underbrace{T^{ij}}_{g^{ik}g^{jl}T^{kl}}$$

$$= \dot{\rho} + 3\frac{\dot{a}}{a}\rho + \dot{a}a \underbrace{\gamma_{ij} g^{ik}}_{-a^{-2}\delta^k_j} g^{jl} \left(-g_{kl}\, p\right)$$

$$\Rightarrow \quad \boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0} \qquad (E - p \text{ conservation})$$

Note: $\rho(t)$ and $p(t)$ related by equation of state (eos) of the fluid:

$$p = p(\rho)$$

$\Rightarrow$ For given eos, evolution of universe fully determined by Friedmann eq. + $E - p$ conservation

ij- component of Einstein equation gives

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G\rho - \frac{k}{a^2} \qquad \text{(automatically satisfied)}$$

4.1 Simple cosmological solutions $(k = 0)$

Example 1: Non-relativistic matter (Einstein- de Sitter Universe)

$$\text{EOS: } p \sim mr^2 \approx 0 \quad \Rightarrow \frac{\dot\rho}{\rho} = 3\frac{\dot a}{a} \Rightarrow \rho = \frac{\text{const}}{a^3}$$

Interpretation: Conservation of particle number N

$$\Lsh\, n = \frac{N}{V} \sim a^{-3} \Rightarrow \rho = m \cdot n \sim a^{-3}$$

Friedmann eq. :

$$\left(\frac{\dot a}{a}\right)^2 = \frac{\text{const}}{a^3} \Rightarrow \sqrt{a}da = \text{const}\, dt$$

$$\Rightarrow a^{\frac{3}{2}} = \text{const.}\,(t - t_s)$$

$$\Rightarrow a(t) = \text{const.}\,(t - t_s)^{\frac{2}{3}}$$

$$H(t) = \frac{\dot a(t)}{a(t)} = \frac{2}{3(t - t_s)}$$

For $t \to t_s$: $a \to 0, H \to \infty$

$\Lsh$ singularity ("Big bang")

$\Lsh$ convention: $t_s = 0$

For $t \to \infty$: $a \to \infty, H \to 0$

$\Lsh$ Universe keeps expanding forever, but expansion slows down

The age of a matter-dominated universe is

$$t_0 = \frac{2}{3H_0} \sim 10^{10}\,\text{yr} \qquad (H_0 \text{ inferred from Hubble's law})$$

Present day density: $\rho_0 = \frac{3}{8\pi G} H_0^2 \approx 10^{-29} \frac{\text{g}}{\text{cm}^3}$

Example 2: Relativistic matter (radiation)

$$T^\mu_\mu = 0 \Longleftrightarrow \rho - 3p = 0$$

$$\Rightarrow \text{EOS: } p = \frac{1}{3}\rho \Rightarrow \frac{\dot\rho}{\rho} = 4\frac{\dot a}{a} \Rightarrow \rho = \frac{\text{const}}{a^4}$$

Interpretation: Energy of each particle redshifts $\sim \frac{1}{a}$

$$\Rsh \rho = E \cdot n \sim \frac{1}{a} \cdot \frac{1}{a^3} = \frac{1}{a^4}$$

Friedmann eq.: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{\text{const}}{a^4} \Rightarrow a(t) = \text{const} \cdot (t)^{\frac{1}{2}} \Rightarrow H(t) = \frac{1}{2t}$

Note: If the radiation has a thermal (black-body) spectrum, we can define its temperature

$$\rho = \frac{\pi^2}{30} g T^4 \qquad (g\text{: degrees of freedom})$$

$$\Rightarrow T \sim \frac{1}{a} \sim (1+z)$$

Useful relation: $\boxed{H = \sqrt{\frac{8\pi^3}{90}} \sqrt{g} \frac{T^2}{M_{\text{pl}}} \approx 1.66 \sqrt{g} \frac{T^2}{M_{\text{pl}}}}$ with $M_{\text{pl}} = G^{-1/2}$

Example 3: Vacuum energy

Assume that vacuum has non-vanishing energy density $T_{\mu\nu} = \rho g_{\mu\nu}$

$$\Rightarrow p = -\rho \qquad \text{(negative pressure)}$$

$$\Rightarrow \dot{\rho} = 0$$

Interpretation: Vacuum energy does not dilute as space expands

Friedmann eq.: $\frac{\dot{a}}{a} = \text{const} \Rightarrow a = \text{const} \cdot e^{Ht}$ with $H = \sqrt{\frac{8\pi}{3} G \rho}$

Very different from examples 1 + 2 :

- All quantities finite for $t \to -\infty \Rightarrow$ No singularity
- $\ddot{a} > 0 \Rightarrow$ Accelerated expansion

Example 4: General component with $p = w\rho \quad (w > -1)$

$$\Rightarrow a = \text{const} \cdot t^{\frac{2}{3(1+w)}} \longrightarrow \begin{array}{l} \text{Decelerated expansion for } w > -1/3 \\ \text{Accelerated expansion for } w < -1/3 \end{array}$$

Comment: Possible to generalize Einstein eq. to

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (\Lambda : \text{cosmological constant})$$

↳ First introduced by Einstein to guarantee stationary universe ($H = 0$)

↳ Hubble's law: $H \neq 0$
Einstein: "Größte Eselei meines Lebens"

Modern interpretation: Λ contributes to vacuum energy

$$\rho_{\text{vac}} \to \rho_{\text{vac}} + \frac{\Lambda}{8\pi G}$$

But: So far no successful prediction of ρ_{vac} from first principles
→ "Cosmological constant problem"

Conventions: $\rho_{\text{vac}} \equiv \rho_\Lambda$
vacuum energy ≡ dark energy

$k = 1 \Leftrightarrow$ closed universe
$k = 0 \Leftrightarrow$ flat universe
$k = -1 \Leftrightarrow$ open universe

Note:

$$T^\nu_\mu = (p+\rho)u_\mu u^\nu - \delta^\nu_\mu p = \begin{pmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix} \Rightarrow \text{same as in Minskowski space}$$

But

$$T_{\mu\nu} = g_{\nu\rho} T^\rho_\mu \begin{pmatrix} \rho & & & \\ & a^2 p & & \\ & & a^2 p & \\ & & & a^2 p \end{pmatrix} \Rightarrow \text{different from Minskowski space}$$

4.2 The ΛCDM model

In general, the energy density of the Universe is a sum of different components

$$\rho_{\text{tot}} = \rho_{\text{M}} + \rho_{\text{rad}} + \rho_\Lambda \qquad \Rightarrow H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\left(\rho_{\text{M}} + \rho_{\text{rad}} + \rho_\Lambda\right) - \frac{k}{a^2}$$

$$\text{Define } \rho_c = \frac{3}{8\pi G} H_0^2 \qquad \text{(critical density)}$$

$$\Omega_i = \frac{\rho_{i,0}}{\rho_c} \qquad \text{(present-day abundance)}$$

$$\Omega_{\text{curv}} = -\frac{k}{a^2\, H_0^2}$$

$$\Rightarrow \quad \Omega_{\text{M}} + \Omega_{\text{rad}} + \Omega_{\Lambda} + \Omega_{\text{curv}} = 1$$

Note: $\Omega_{\text{M}} + \Omega_{\text{rad}} + \Omega_{\Lambda} = 1 \Leftrightarrow \rho_{\text{tot}} = \rho_c \Longleftrightarrow k = 0$ (flat universe)

Observations yield $\Omega_{\text{M}} + \Omega_{\Lambda} \approx 1 \Rightarrow \Omega_{\text{rad}}, \Omega_{\text{curv}} \ll 1$

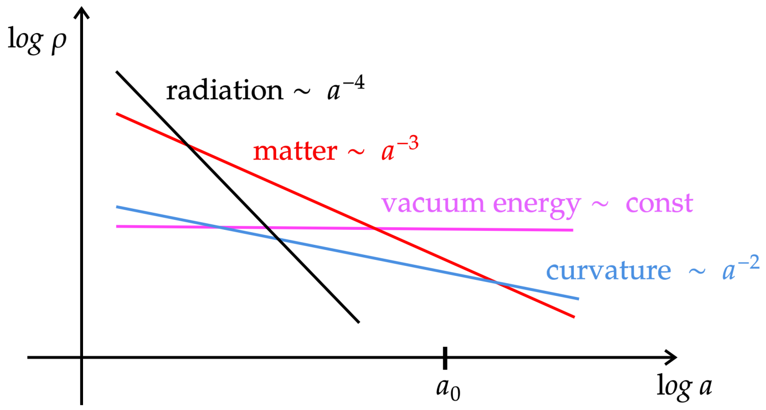
Lower bound on Ω_{rad} from CMB:

$$\rho_{\text{rad},0} \geq \rho_{\gamma,0} = 2\frac{\pi^2}{30} {T_0}^4 = 2.6 \cdot 10^{-10}\, \frac{\text{GeV}}{\text{cm}^3} \qquad (T_0 \approx 2.726\,\text{K})$$

$$\rho_c \approx 5 \cdot 10^{-6}\, \frac{\text{GeV}}{\text{cm}^3} \quad \Rightarrow \quad \Omega_{\text{rad}} \gtrsim 5 \cdot 10^{-5}$$

Using $\rho_{\text{M}} \sim a^{-3}$, $\rho_{\text{rad}} \sim a^{-4}$, $\rho_{\Lambda} \sim \text{const}$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\, \rho_c \Bigg[\underbrace{\Omega_{\text{M}} \left(\frac{a_0}{a}\right)^3 + \Omega_{\text{rad}} \left(\frac{a_0}{a}\right)^4}_{\text{dominate for small } a} + \underbrace{\Omega_{\Lambda} + \Omega_{\text{curv}} \left(\frac{a_0}{a}\right)^2}_{\text{dominate for large } a} \Bigg]$$



For $a \approx a_0$, we can neglect $\Omega_{\rm rad}$, $\Omega_{\rm curv}$

$$\Rightarrow \dot{a}^2 = \frac{8\pi}{3} G \rho_c \left(\Omega_{\rm M} \frac{{a_0}^3}{a} + \Omega_\Lambda a^2 \right) \qquad (\star)$$

$$\ddot{a} = a \frac{4\pi}{3} G \rho_c \left(2\Omega_\Lambda - \Omega_{\rm M} \left(\frac{a_0}{a} \right)^3 \right)$$

Transition from decelerated ($\ddot{a} < 0$) to accelerated ($\ddot{a} > 0$) expansion at

$$\left(\frac{a_0}{a_{ac}} \right)^3 = \frac{2\Omega_\Lambda}{\Omega_{\rm M}}$$

Realistic values ($\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$)

$$z_{ac} = \left(\frac{2\Omega_\Lambda}{\Omega_{\rm M}} \right)^{1/3} - 1 \approx 0.76 \qquad \text{(pretty recent!)}$$

For $a \ll a_0$, we can neglect Ω_Λ, $\Omega_{\rm curv}$

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \left(\frac{a_0}{a} \right)^3 \left[\Omega_{\rm M} + \Omega_{\rm rad} \frac{a_0}{a} \right]$$

$$\Rightarrow \text{Matter-radiation equality: } \frac{a_0}{a_{eq}} = \frac{\Omega_{\rm M}}{\Omega_{\rm rad}} \sim 10^4$$

More accurate estimate: $\rho_{\rm rad} = \rho_\gamma + \rho_\nu \approx 1.7\, \rho_\gamma$ (will derive this later!)

$$\Rightarrow 1 + z_{\rm eq} = \frac{a_0}{a_{\rm eq}} \approx 3 \cdot 10^3$$

$$T_{\rm eq} = (1 + z_{\rm eq})\, T_0 \approx 0.75\, \text{eV}$$

For $a < a_{eq}$, universe is radiation dominated

$$\Rightarrow t_{\rm eq} \approx \frac{1}{2H_{\rm eq}} \approx 7 \cdot 10^4\, \text{yr}$$

For $a \gg a_{\rm eq}$:

$$(\star) \Rightarrow \quad a(t) = a_0 \left(\frac{\Omega_M}{\Omega_\Lambda} \right)^{1/3} \sinh^{2/3} \left(\frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right)$$

$$\sim \begin{cases} t^{2/3} & \text{for } \frac{3}{2}\sqrt{\Omega_\Lambda} H_0 t \ll 1 \\ e^{\sqrt{\Omega_\Lambda} H_0 t} & \text{for } \frac{3}{2}\sqrt{\Omega_1} H_0 t \gg 1 \end{cases}$$

$$\left(\frac{a_0}{a\left(t_{\rm ac}\right)}\right)^3 = \frac{2\Omega_\Lambda}{\Omega_{\rm M}} \Rightarrow t_{\rm ac} \sim 7.3\,{\rm Gyr}$$

$$\frac{a_0}{a\left(t_0\right)} = 1 \Rightarrow t_0 \sim 13.5\,{\rm Gyr}$$

Because of recent accelerated expansion, age of Universe is larger than for $\Omega_{\rm M} = 1$

↳ Consistent with observation of ancient stars requiring $t_0 > 13\,{\rm Gyr}$
$\Rightarrow$ Evidence for $\Omega_\Lambda > 0$

How to obtain more accurate estimates of Ω_i?

4.3 Brightness-redshift relation

To measure expansion history, need far-away objects of known absolute luminosity ("standard candles")

Example: Type Ia supernovae (SNe Ia)

↳ Thermonuclear explosion of a white dwarf in a binary system

↳ Known relation between peak luminosity and time-dependence of emission

Need to relate absolute luminosity $L = \frac{\text{emitted energy}}{\text{time}}$ to observed brightness $\mathcal{J}$

$$\mathcal{J} = \frac{\#\text{ photons} \cdot \text{observed energy}}{\text{time} \cdot \text{area}}$$

Consider photons emitted at t_i and observed at t_0.

$$\text{Observed energy} = \text{emitted energy} \cdot \frac{a(t_i)}{a_0}$$

$$\frac{\#\text{ photons}}{\text{time}} = \frac{\#\text{ emitted photons}}{\text{time}} \cdot \frac{a(t_i)}{a_0} \qquad \frac{a(t_i)}{a_0}: \text{redshift}$$

To calculate the area, use

$$ds^2 = dt^2 - a(t)^2\left[d\chi^2 + S_k^2(\chi)d\Omega^2\right] = 0 \qquad S_k(\chi) = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$$

$$\Rightarrow \chi\left(t_i\right) = \int_{t_i}^{t_0} \frac{dt}{a(t)}$$

$$z(t) = \frac{a_0}{a(t)} - 1 \quad \Rightarrow \quad dz = -\frac{a_0}{a(t)^2}\dot{a}(t)dt = -\frac{a_0}{a(t)}H(z)dt$$

$$\chi(z) = \int_0^z \frac{dz'}{a_0 H(z')} \approx \int_0^z \frac{dz'}{a_0 H_0} \frac{1}{\sqrt{\Omega_M (z'+1)^3 + \Omega_\Lambda + \Omega_{curv} (z'+1)^2}}$$

At $t = t_0$ the photons pass through a sphere of size $S(z) = 4\pi \underline{d}^2(z) = 4\pi {a_0}^2 S_k^2(\chi(z))$

$$\Rightarrow \boxed{\mathcal{J} = \frac{L}{(1+z)^2 S(z)} = \frac{L}{4\pi {r_L}^2}} \quad \text{with } r_L = (1+z)\, a_0\, S_k(\chi(z)) \quad \text{("luminosity distance")}$$

Comment:

- For $z \ll 1 : (z'+1) \approx 1 \Rightarrow \chi(z) \approx \frac{z}{a_0 H_0} \Rightarrow \underline{d(z)} \approx \frac{z}{H_0}$ (Hubble's law)
- Consider $\Omega_{curv} = 0 \Rightarrow \Omega_M + \Omega_\Lambda = 1$

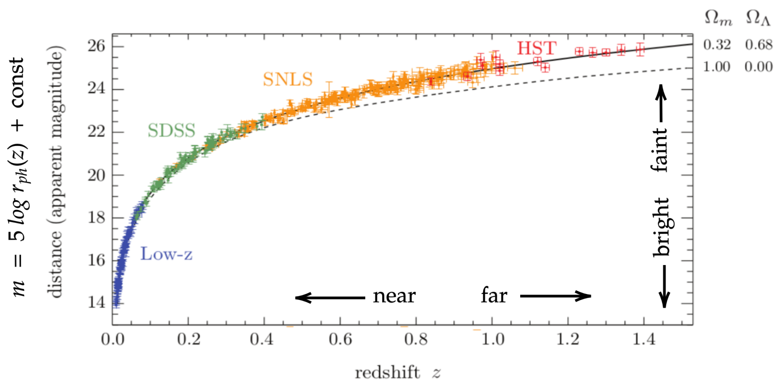
$$\Rightarrow H_0\, \underline{d(z)} = \int_0^z \frac{dz'}{\sqrt{\Omega_M (z'+1)^3 + (1-\Omega_M)}}$$

$$= \int_0^z \frac{dz'}{\sqrt{\Omega_M (3z' + 3z'^2 + z'^3) + 1}}$$

$$= \begin{cases} 2\left(1 - \frac{1}{\sqrt{1+z}}\right) & \Omega_M = 1, \Omega_\Lambda = 0 \\ z & \Omega_M = 0, \Omega_\Lambda = 1 \end{cases}$$

$\Rightarrow$ $\underline{d(z)}$ increases with decreasing Ω_M

$\Rightarrow$ $\mathcal{J}$ decreases with decreasing Ω_M
↳ vacuum energy makes standard candles less bright



Exactly what is observed! → Nobel prize 2011

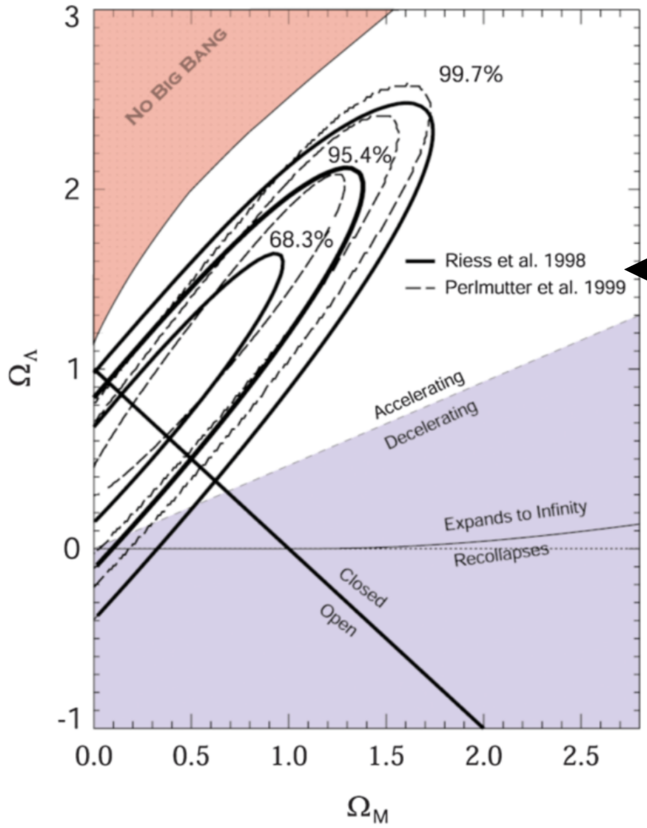
- For $\Omega_{\mathrm{curv}} > 0 \quad (k = -1)$ we obtain

$$\begin{aligned}
\chi(z) &\overset{z \ll 1}{\approx} \int_0^z \frac{dz'}{a_0 H_0} \frac{1}{\sqrt{(1+3z')\ \Omega_{\mathrm{M}} + \Omega_\Lambda + (1+2z')\ \Omega_{\mathrm{curv}}}} \\
&\approx \int_0^z \frac{dz'}{a_0 H_0} \left(1 - \frac{1}{2} \left(3\,\Omega_{\mathrm{M}} + 2\,\Omega_{\mathrm{curv}}\right) z' \right) \\
&= \frac{1}{a_0 H_0} \left(z - \frac{z^2}{4} \left(3\,\Omega_{\mathrm{M}} + 2\,\Omega_{\mathrm{curv}}\right) + \mathcal{O}\left(z^3\right) \right) \\
&= \frac{1}{a_0 H_0} \left(z - \frac{z^2}{4} \left(2 + \Omega_{\mathrm{M}} - 2\,\Omega_\Lambda\right) + \mathcal{O}\left(z^3\right) \right)
\end{aligned}$$

↳ Non-linear correction to Hubble's law

Data clearly requires $\Omega_{\mathrm{M}} - 2\,\Omega_\Lambda > 0$

⇒ Present universe experiences accelerated expansion!



very old data!
modern constraints
much tighter!

5 Early Universe Thermodynamics

So far: Treated matter and radiation as non-interacting perfect fluids

More realistic: Ensembles of interacting particles

↳ Sufficiently strong interactions ⇒ local thermal equilibrium (LTE) (will quantify this next lecture!)

Each particle species i characterised by distribution function

$$f_i(\vec{p}) = \frac{1}{(2\pi)^3} \frac{1}{e^{(E_i - \mu_i)/T} \mp 1} \qquad \begin{array}{ll} - : & \text{boson} \\ + : & \text{fermion} \end{array}$$

with $E_i = \sqrt{\vec{p}^2 + m_i^2}$

T : temperature (common for all species)

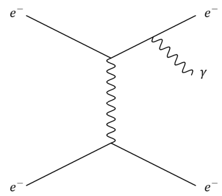
μ_i : chemical potential (may depend on T)

For process $A_1 + A_2 + \ldots \longleftrightarrow B_1 + B_2 + \ldots$ in chemical equilibrium:

$$\mu_{A_1} + \mu_{A_2} + \ldots = \mu_{B_1} + \mu_{B_2} + \ldots$$

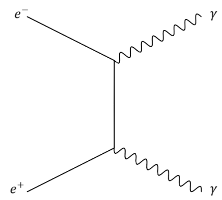
Examples: $e^- + e^- \to e^- + e^- + \gamma$

$\Rightarrow \mu_\gamma = 0$



$e^- + e^+ \to 2\gamma$

$\Rightarrow \mu_{e^+} = -\mu_{e^-}$



For $m_i \gg T, \mu_i$: $\quad E_i \approx m_i + \frac{1}{2}\frac{\vec{p}^2}{m_i}$

$$\Rightarrow f_i(\vec{p}) \approx \frac{1}{(2\pi)^3} e^{(\mu_i - m_i)/T} e^{-\vec{p}^2/2m_i T}$$

For given $f_i(\vec{p})$, we can calculate

- number density

$$n_i = g_i \int f_i(\vec{p}) d^3p \overset{EdE=pdp}{=} 4\pi g_i \int f_i(E) \sqrt{E^2 - m_i^2}\, EdE$$

- energy density

$$\rho_i = g_i \int f_i(\vec{p}) E_i(\vec{p}) d^3p = 4\pi g_i \int f_i(E) \sqrt{E^2 - m_i^2}\, E^2 dE$$

- pressure

$$p_i = \frac{g_i}{3} \int f_i(\vec{p}) \frac{\vec{p}^2}{E_i(\vec{p})} d^3p = \frac{4\pi g_i}{3} \int f_i(E) \left(E^2 - m_i^2\right)^{3/2} dE$$

g_i : degrees of freedom

SM:

i	$=$	γ	e^-	e^+	Z	W^-	W^+	ν	$\bar{\nu}$	h	q	$\bar{q}$	g
g_i	$=$	2	2	2	3	3	3	1	1	1	6	6	16

$$\sum_i g_i = \sum_{\text{bosons}} g_i + \sum_{\text{fermions}} g_i = (2 + 3 \times 3 + 1 + 16) + (3 \times 2 \times (2 + 1 + 6 + 6))$$

$$= 28 + 90 = 118$$

5.1 Relativistic species

Assume $T \gg m_i, \quad \mu_i = 0$

$$\Rightarrow \rho_i = \frac{g_i}{2\pi^2} \int \frac{E^3}{e^{E/T} \mp 1} dE = \begin{cases} g_i \cdot \frac{\pi^2}{30} T^4 & \text{boson} \\ \frac{7}{8} g_i \cdot \frac{\pi^2}{30} T^4 & \text{fermion} \end{cases}$$

For several relativistic species

$$\boxed{\rho = \sum_i \rho_i = g_* \frac{\pi^2}{30} T^4 \qquad \text{with } g_* = \sum_{\text{rel. bosons}} g_i + \frac{7}{8} \sum_{\text{rel.fermions}} g_i}$$

(effective number of rel. degrees of freedom)

Examples:

- $T \gg m_t: \quad g_* = 28 + \frac{7}{8} \cdot 90 = 106.75$
- $m_\mu \gg T \gg m_e: \quad g_* = 2 + \frac{7}{8}(2 \times 2 + 3 \times 2 \times 1) = 10.75$

$$p_i = \frac{g_i}{6\pi^2} \int \frac{E^3}{e^{E/T} \mp 1} dE = \frac{\rho_i}{3} \qquad \text{(as expected)}$$

$$n_i = \frac{g_i}{2\pi^2} \int \frac{E^2}{e^{E/T} \mp 1} dE = \begin{cases} g_i \cdot \frac{\zeta(3)}{\pi^2} T^3 & \text{boson} \\ \frac{3}{4} g_i \cdot \frac{\zeta(3)}{\pi^2} T^3 & \text{fermion} \end{cases} \qquad \zeta(3) \approx 1.2$$

$$\Rightarrow \langle E \rangle = \frac{\rho_i}{n_i} = \begin{cases} 2.70\,T & \text{boson} \\ 3.15\,T & \text{fermion} \end{cases}$$

5.2 Non-relativistic species

$$n_i = \frac{g_i}{2\pi^2} e^{\frac{\mu_i - m_i}{T}} \int \underbrace{e^{-p^2/2m_i T} p^2}_{\substack{\text{Maxwell-} \\ \text{Boltzmann} \\ \text{distribution}}} dp = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{\frac{\mu_i - m_i}{T}}$$

For $\mu_i = 0$, density of non-rel particles is exponentially suppressed (Boltzmann suppression)

Interpretation: Annihilation process $\underset{\text{heavy}}{h + h} \to \underset{\text{light}}{l + l}$ always possible

Production process $l + l \to h + h$ requires $E > 2m_h$

↳ Exponentially unlikely for $T \ll m_h$

↳ Heavy particles "annihilate away"

↳ Energy density dominated by light species (unless $\mu_i \neq 0$)

$$\rho_i = m_i \cdot n_i + \frac{3}{2} n_i T \overset{T \to 0}{\approx} m_i n_i$$

$$p_i = T n_i \overset{T \to 0}{\approx} 0 \qquad \text{(Note: } pV = NT \text{ ideal gas law for } k_B = 1\text{)}$$

5.3 Entropy

First law of thermodynamics: $dE = TdS - pdV + \sum_i \mu_i dN_i$

Define $s = \frac{S}{V}$ (entropy density)

$$\Rightarrow ds = \frac{dS}{V} - s\frac{dV}{V} \qquad \text{(analogous for } \rho = \frac{E}{V},\, n = \frac{N}{V}\text{)}$$

$$\Rightarrow \left(Ts - p - \rho + \sum_i \mu_i n_i\right) dV + (Tds - d\rho + \mu dn)V = 0$$

Consider $V = \text{const.}$ $\quad \Rightarrow dV = 0 \Rightarrow Tds - d\rho + \mu dn = 0$

For arbitrary volume $\quad \Rightarrow Ts - p - \rho + \sum_i \mu_i n_i = 0$

$$\Rightarrow \boxed{s = \frac{p + \rho - \sum_i \mu_i n_i}{T}}$$

Example:

- Rel. species with $\mu_i = 0$

$$\Rightarrow s_i = \frac{p_i + \rho_i}{T} = \frac{4}{3}\frac{\rho_i}{T} = \begin{cases} g_i \frac{2\pi^2}{45} T^3 & \text{boson} \\ \frac{7}{8} g_i \frac{2\pi^2}{45} T^3 & \text{fermion} \end{cases}$$

$$\Rightarrow s = \sum_i s_i = g_* \frac{2\pi^2}{45} T^3$$

- Non-rel species

$$\begin{aligned} s_i &= \frac{\rho_i + p_i - \mu_i n_i}{T} \\ &= \frac{m_i n_i + \frac{3}{2} n_i T + n_i T - \mu_i n_i}{T} \\ &= n_i \left(\frac{5}{2} + \frac{m_i - \mu_i}{T}\right) \qquad \frac{m_i - \mu_i}{T} = \log\left[\frac{g_i}{n_i}\left(\frac{m_i T}{2\pi}\right)^{3/2}\right] \end{aligned}$$

$\Rightarrow$ Similar Boltzmann suppression as for n_i

Second law of thermodynamics: $dS = 0$ for equilibrium evolution

Proof (assuming $\sum_i \mu_i dn_i = 0$):

$$TdS = pdV + d(\rho \cdot V) = (p + \rho)dV + Vd\rho$$

Remember: $V \sim a^3 \Rightarrow dV = 3a^2 da = 3V \frac{da}{a}$

$$\Rightarrow T\frac{dS}{dt} = V \underbrace{\left(3(p+\rho)\frac{\dot{a}}{a} + \dot{\rho}\right)}_{=0\,(E-p \text{ conservation})}$$

$\Rightarrow s \cdot a^3 = \text{const}$

$\Rightarrow$ entropy density convenient measure of expansion

Define $Y_i = \frac{n_i}{s} \sim n_i \cdot V = N_i$

If no particles are produced/destroyed $\Rightarrow Y_i = \text{const}$

Examples: Baryon number conservation: $N_B - N_{\bar{B}} = \text{const}$

$$\Rightarrow \Delta_B = \frac{n_B}{s} - \frac{n_{\bar{B}}}{s} = \text{const}$$

Particle thresholds

Shown before that $T \sim a^{-1}$ during RD

Implicitly assumed $g_* = \text{const}$

More accurate: $g_* T^3 a^3 = \text{const} \Rightarrow T \sim {g_*}^{-1/3} a^{-1}$

If T drops below m_i

$\Rightarrow$ species becomes non-relativistic

$\Rightarrow$ g_* decreases

$\Rightarrow$ T decreases more slowly

Interpretation: As non-relativistic particles annihilate away, entropy transferred to relativistic species

6 Boltzmann equation

Last time: Assumed all species to be in equilibrium

↳ Not always satisfied

↳ Departure from equilibrium essential for cosmology

Today: General evolution of $f(p,t) \leftarrow$ homogeneous and isotropic

$$\underbrace{L[f]}_{\substack{\text{Liouville operator} \\ \to \text{ phase space evolution}}} = \underbrace{C[f]}_{\substack{\text{Collision operator} \\ \to \text{ effect of interactions}}}$$

For $C[f]=0$: Particle number conserved

$\Rightarrow$ Phase space volume conserved

$$\Rightarrow \frac{\mathrm{d}f(p,t)}{\mathrm{d}t} = 0 = \frac{\partial f}{\partial t} + \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\partial f}{\partial p}$$

Consider particle 4-momentum $P^\mu = (E, \vec{p})$

$$pdp = EdE = P^0 dP^0$$

$$\Rightarrow p\frac{\mathrm{d}p}{\mathrm{d}t} = P^0 \frac{\mathrm{d}P^0}{\mathrm{d}t} \underset{\substack{\text{geodesic} \\ \text{eq.}}}{=} -\Gamma^0_{\alpha\beta} P^\alpha P^\beta = -H(t)p^2$$

$$\Rightarrow L[f] = \frac{\partial f}{\partial t} - H(t)\, p\, \frac{\partial f}{\partial p} = \frac{\partial f}{\partial t} - H(t)\frac{p^2}{E}\frac{\partial f}{\partial E}$$

Often convenient to consider integral

$$\frac{g}{(2\pi)^3}\int d^3p\, L[f] = \frac{\partial}{\partial t}\underbrace{\left(\frac{g}{(2\pi)^3}\int d^3p f\right)}_{=n(t)} - H\frac{\partial}{(2\pi)^3}\underbrace{\int d^3p\, p\frac{\partial f}{\partial p}}_{=-4\pi\int_0^\infty 3p^2 f dp}$$

$$= \dot{n} + 3Hn = \frac{1}{a^3}\frac{d}{dt}\left(na^3\right)$$

↳ Liouville operator describes change in $n(t)$ due to expansion

To find explicit form for $C[f]$ consider interaction $1+2 \leftrightarrow 3+4$

↳ Decrease in f_1 proportional to

$$\sum_{\text{spins}} \underbrace{|\mathcal{M}_{12\to 34}|^2}_{\substack{\text{reaction} \\ \text{probability}}} \cdot \underbrace{f_3 f_4}_{\substack{\text{density of} \\ \text{initial states}}} \cdot \underbrace{(1\pm f_1)(1\pm f_2)}_{\substack{\text{boson: + (Bose enhancement)} \\ \text{fermion: - (Pauli blocking)}}}$$

↳ Increase in f_1 proportional to

$$\sum_{\text{spins}} \left|\mathcal{M}_{34\to 21}\right|^2 f_1 f_2 \left(1 \pm f_3\right)\left(1 \pm f_4\right)$$

Simplifications:

- Often possible to assume $f \ll 1 \Rightarrow (1 \pm f) \approx 1$
- For most processes $\left|\mathcal{M}_{12\to 34}\right|^2 = \left|\mathcal{M}_{34\to 21}\right|^2 \equiv \left|\mathcal{M}\right|^2$
- Need to integrate over all possible momenta

$$\Rightarrow C\left[f_1\right] = \frac{1}{2E_1} \int \underset{\left(d\Pi = \frac{d^3 p}{(2\pi)^3 2E}\right)}{d\Pi_2 d\Pi_3 d\Pi_4} \; (2\pi)^4 \delta^4\left(p_1 + p_2 - p_3 - p_4\right) \times \sum_{\text{spins}} \left|\mathcal{M}\right|^2 \left(f_3 f_4 - f_1 f_2\right)$$

Additional assumption: f_3 and f_4 given by equil. dist.

$$\Rightarrow f_3 \cdot f_4 = f_3^{\text{eq}} \cdot f_4^{\text{eq}} = e^{-(E_3+E_4)/T}$$
$$\overset{E \text{ cons.}}{=} e^{-(E_1+E_2)/T} = f_1^{\text{eq}} f_2^{\text{eq}}$$

$$\Rightarrow C\left[f_1\right] = \frac{1}{2E_1} \int d\Pi_2 \left(f_1^{\text{eq}} f_2^{\text{eq}} - f_1 f_2\right) \times \underbrace{\int d\Pi_3 d\Pi_4 (2\pi)^4 \delta^4\left(p_1 + p_2 - p_3 - p_4\right) \sum_{\text{spins}} \left|\mathcal{M}\right|^2}_{=\sigma v}$$

$$\Rightarrow \frac{g}{(2\pi)^3} \int d^3 p \, C\left[f_1\right] = \int d\Pi_1 d\Pi_2 \left(f_1^{\text{eq}} f_2^{\text{eq}} - f_1 f_2\right) \sigma v$$

Now assume $\dfrac{n}{n^{\text{eq}}} = \dfrac{f}{f^{\text{eq}}}$

$$\Rightarrow \frac{g}{(2\pi)^3} \int d^3 p \, C\left[f_1\right] = \left(n_1^{\text{eq}} n_2^{\text{eq}} - n_1 n_2\right) \times \underbrace{\frac{1}{n_1^{\text{eq}} n_2^{\text{eq}}} \int d\Pi_1 d\Pi_2 f_1^{\text{eq}} \rho_2^{\text{eq}} \sigma v}_{\substack{\equiv \langle \sigma v \rangle \\ \text{“thermally averaged cross section”}}}$$

$$\Rightarrow \quad \boxed{\dot{n}_1 + 3 H n_1 = \langle \sigma v \rangle \left(n_1^{\text{eq}} n_2^{\text{eq}} - n_1 n_2\right)} \qquad \text{(Boltzmann equation)}$$

Example: Consider $e^+e^- \leftrightarrow \gamma\gamma$ with $n_{e^+} = n_{e^-} \equiv n$

$$\Rightarrow \dot{n} + 3Hn = \langle\sigma v\rangle \left[(n^{\text{eq}})^2 - n^2\right]$$

Using $Y = \frac{n}{s}$ and $3Hs + \dot{s} = 0$

$$\Rightarrow \dot{n} = \dot{Y}s + Y\dot{s} = \dot{Y}s - 3HsY$$

$$\Rightarrow \dot{Y} = -\langle\sigma v\rangle s \left(Y^2 - Y_{\text{eq}}^2\right)$$

Define $x \equiv \frac{m}{T} \Rightarrow \frac{ds}{dx} = \dot{s}\frac{dt}{dx} = -3H\,s\frac{dt}{dx}$

$$\Rightarrow \frac{dY}{dx} = \frac{1}{3H}\frac{ds}{dx}\langle\sigma v\rangle \left(Y^2 - Y_{\text{eq}}^2\right)$$

For $g_* = \text{const}$: $s \sim T^3$

$$\Rightarrow \frac{ds}{dx} = 3\frac{s}{T}\frac{dT}{dx} = -\frac{3s}{x}$$

$$\Rightarrow \boxed{\frac{dY}{dx} = -\frac{s}{Hx}\langle\sigma v\rangle \left(Y^2 - Y_{\text{eq}}^2\right)}$$

Interpretation: $\sigma \cdot \underbrace{v \cdot n}_{\text{particle flux}} = \Gamma$ "interaction rate"

↳ determines how rapid an interaction happens

$$\Rightarrow \underbrace{\frac{x}{Y_{\text{eq}}}\frac{dy}{dx}}_{\text{rel. change in density}} = -\underbrace{\frac{\Gamma}{H}}_{\text{interaction}} \cdot \underbrace{\left(\frac{Y^2}{Y_{\text{eq}}^2} - 1\right)}_{\text{departure from equil.}}$$

- If $Y < Y_{\text{eq}}$: $\frac{dY}{dx} > 0 \Rightarrow$ increase
- If $Y > Y_{\text{eq}}$: $\frac{dY}{dx} < 0 \Rightarrow$ decrease

} evolution towards equilibrium

- For $\Gamma \gg H$: Quick evolution $Y \to Y_{\text{eq}}$ $\Rightarrow$ thermal equilibrium
- For $\Gamma \ll H$: $\frac{dY}{dx} \to 0 \Rightarrow$ no thermal equilibrium

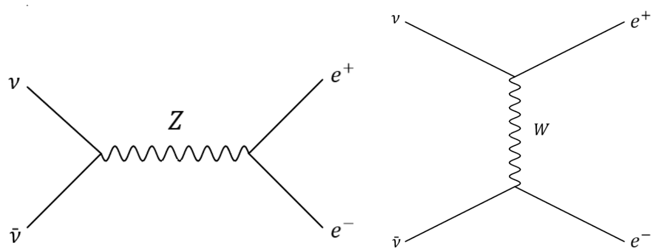
$\Rightarrow$ comoving number density constant

For $e^+e^- \leftrightarrow \gamma\gamma$ and $E \gg m_e$: $\sigma \sim \frac{\alpha^2}{E^2}$, $\langle\sigma v\rangle \sim \frac{\alpha^2}{T^2}$ and $n_{\text{eq}} \sim T^3 \Rightarrow \Gamma \sim \alpha^2 T$

$$\Rightarrow \quad \frac{\Gamma}{H} \sim \frac{\alpha^2 M_\mathrm{P}}{T} \sim \frac{10^{15}\,\mathrm{GeV}}{T} \gg 1 \quad \text{for all relevant } T$$

$\Rightarrow$ e^+, e^-, γ in thermal equilibrium
↳ Also true for other particles with EM charge

Next: Consider $\nu\bar{\nu} \leftrightarrow e^+e^-$



For $m_e \ll E \ll m_{W,Z}$: $\quad \sigma \sim G_F^2 E^2 \qquad (G_F = 1.17 \cdot 10^{-5}\,\mathrm{GeV}^{-2})$

$$\Rightarrow \quad \langle \sigma v \rangle \sim G_F^2 T^2$$

$$\Rightarrow \quad \Gamma \sim G_F^2 T^5$$

$$\Rightarrow \quad \frac{\Gamma}{H} \sim G_F^2 T^3 M_\mathrm{P} \sim 10^9\,\mathrm{GeV}^{-3}\, T^3 \sim \left(\frac{T}{1\,\mathrm{MeV}}\right)^3$$

$$\Rightarrow \quad \frac{\Gamma}{H} > 1 \Leftrightarrow T > 1\,\mathrm{MeV}$$

$\Rightarrow$ Neutrinos decouple from thermal equilibrium when T drops below $1\,\mathrm{MeV}$
More detailed calculation: $T_\nu^\mathrm{dec} = 2 - 3\,\mathrm{MeV}$ (depends on flavour)

7 Relic neutrinos

Neutrinos decouple from thermal bath at $T_{\text{dec}} \sim 2-3$ MeV. What happens then?

Without interactions, the coordinate momentum $k = a \cdot p$ of each neutrinos is time independent

$$\Rightarrow \quad f(p,t) = f(k) = f_{\text{dec}}\left(\frac{a(t)}{a_{\text{dec}}}\,p\right) \qquad (a_{\text{dec}}: \text{ scale factor at decoupling})$$

$$\text{with} \quad f_{\text{dec}}(p) = \frac{1}{(2\pi)^3}\frac{1}{e^{p/T_{\text{dec}}}+1}$$

$$\Rightarrow \quad f(p,t) = \frac{1}{(2\pi)^3}\frac{1}{e^{(p/T_{\text{eff}}(t))}+1} \quad \text{with} \quad T_{\text{eff}}(t) = \frac{a_{\text{dec}}}{a(t)}\,T_{\text{dec}}$$

↳ Neutrinos maintain thermal distribution even without interactions

↳ Effective temperature decreases as $T_{\text{eff}} \sim a^{-1} \Rightarrow n_\nu \sim a^{-3}$

↳ True even for $T_{\text{eff}} < m_\nu$

⇒ Neutrinos have rel. distribution even in present universe ("hot relic")

⇒ Very different from equilibrium distribution (i.e. Maxwell-Boltzmann)

Since $T_{\text{dec}} > m_e$, there are still many e^+, e^- in thermal bath when neutrinos decouple

↳ Annihilate away for $T \ll m_e : e^+e^- \to \gamma\gamma$

↳ Energy and entropy transferred to photons, but not to neutrinos

⇒ T_γ does not decrease as $a(t)^{-1}$

⇒ For $T_\gamma \ll m_e : T_\gamma \neq T_{\nu,\text{eff}}$

Make use of entropy conservation in electron-photon system:

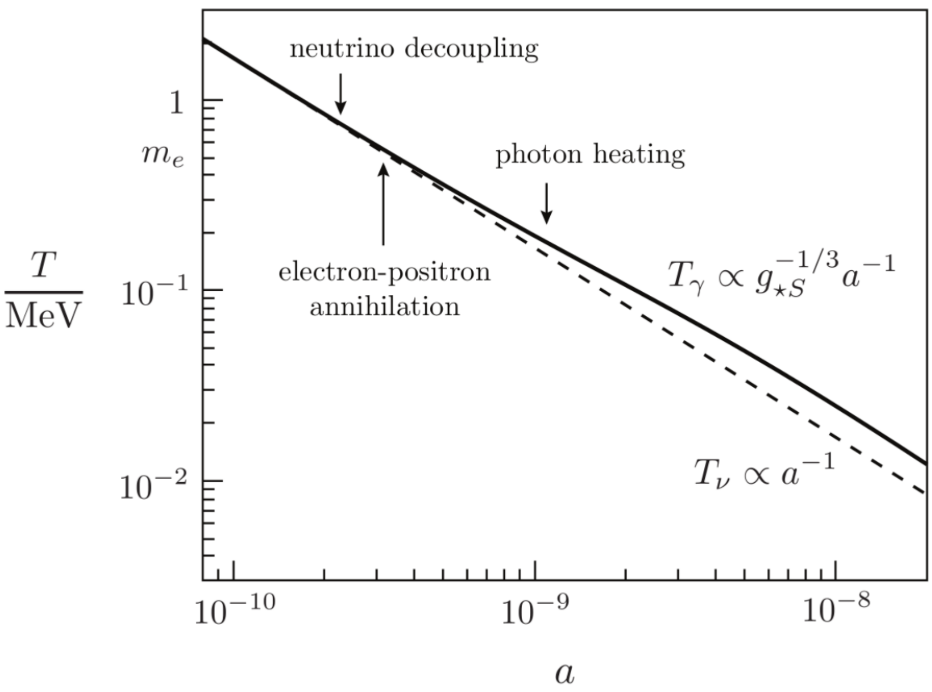
$$g_*^{e\gamma}(T_\gamma)\, a^3 T_\gamma^3 = \text{const}$$

For $T_{\text{dec}} > T_\gamma > m_e$: $\qquad g_*^{e\gamma} = \underset{\gamma}{2} + \frac{7}{8}(\underset{e^-}{2} + \underset{e^+}{2}) = \frac{11}{2}$

For $T_\gamma \ll m_e$: $\qquad g_*^{e\gamma} = 2$

$$\Rightarrow \frac{11}{2} a_{\text{dec}}^3 T_{\text{dec}}^3 = 2\, a^3\, T_\gamma^3 \Rightarrow T_\gamma = \left(\frac{11}{4}\right)^{1/3} T_{\text{dec}}\frac{a_{\text{dec}}}{a}$$

$$\Rightarrow T_\gamma = \left(\frac{11}{4}\right)^{1/3} T_{\nu,\text{eff}}$$



7.1 Neutrinos in the present Universe

Use CMB temperature $T_{\gamma,0} = 2.73$ K to predict present-day temperature of cosmic neutrino background (CνB)

$$T_{C\nu B} = T_{\nu,\text{eff},0} \approx 1.95\,\text{K}$$

$$\Rightarrow n_{\nu,0} = \frac{3}{4} \cdot 2 \cdot \frac{\zeta(3)}{\pi^2} T^3_{C\nu B} \approx 112 \text{ cm}^{-3} \text{ per species}$$

So far not detected. Promising idea: PTOLEMY

$\nu_e + {}^3\text{H} \to {}^3\text{He} + e^-$

↳ Tiny energy release, very challenging!

What about indirect effects?

↳ Contribution of ρ_ν modifies expansion rate

$$\rho_{\nu,0} = \sum m_\nu \cdot n_{\nu,0}$$

↳ Require $\Omega_\nu = \frac{\rho_{\gamma,0}}{\rho_c} < 1 \Rightarrow \sum m_\nu < 50$ eV

Require $\Omega_\nu < \Omega_\text{M} \quad \Rightarrow \sum m_\nu < 15$ eV

KATRIN experiment: $m_{\nu_e} < 0.8$ eV $\Rightarrow \sum m_\nu < 2.4$ eV $\Rightarrow \Omega_\nu < 0.05$

Neutrinos cannot be all of dark matter!

Neutrino oscillation experiments: $\sum m_\nu > 0.06$ eV $\Rightarrow \Omega_\nu > 10^{-3} \gg \Omega_{\text{rad}}$

7.2 Neutrinos during radiation domination

General treatment: Consider non-interacting rel.species with $T_{n_i} \neq T_\gamma$

$$\begin{aligned}
\rho &= \rho_{\text{eq}} + \rho_{n_i} \\
&= \left(\sum_{\substack{\text{bosons}\\ \text{in eq.}}} g_i + \frac{7}{8} \sum_{\substack{\text{fermions}\\ \text{in eq.}}} g_i \right) \frac{\pi^2}{30} T_\gamma^4 + \left(\frac{7}{8}\right) g_{n_i} \frac{\pi^2}{30} T_{n_i}^4 \\
&= g_* \frac{\pi^2}{30} T_\gamma^4 \\
\text{with } g_* &= \sum_{\substack{\text{bosons}\\ \text{in eq.}}} g_i + \frac{7}{8} \sum_{\substack{\text{fermions}\\ \text{in eq.}}} g_i + \left(\frac{7}{8}\right) g_{n_i} \frac{\pi^2}{30} \left(\frac{T_{n_i}}{T_\gamma}\right)^4
\end{aligned}$$

Analogous:

$$\begin{aligned}
s &= g_{*,s} \frac{2\pi^2}{45} T_\gamma^3 \\
\text{with } g_* &= \sum_{\substack{\text{bosons}\\ \text{in eq.}}} g_i + \frac{7}{8} \sum_{\substack{\text{fermions}\\ \text{in eq.}}} g_i + \left(\frac{7}{8}\right) g_{n_i} \frac{\pi^2}{30} \left(\frac{T_{n_i}}{T_\gamma}\right)^3 \\
g_* &\neq g_{*,s} \quad \text{in general}
\end{aligned}$$

In our case: $T_{n_i} = T_{\nu,\text{eff}} = \left(\frac{4}{11}\right)^{1/3} T_\gamma$

$\Rightarrow$ For $T_\gamma < m_e$: $g_* = 2 + 6 \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{4/3} = 3.36$

$$g_{*,s} = 2 + 6 \cdot \frac{7}{8} \cdot \frac{4}{11} = 3.91$$

Convenient to define

$$\boxed{N_{\text{eff}} = \frac{\rho_\nu}{\frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma}}$$

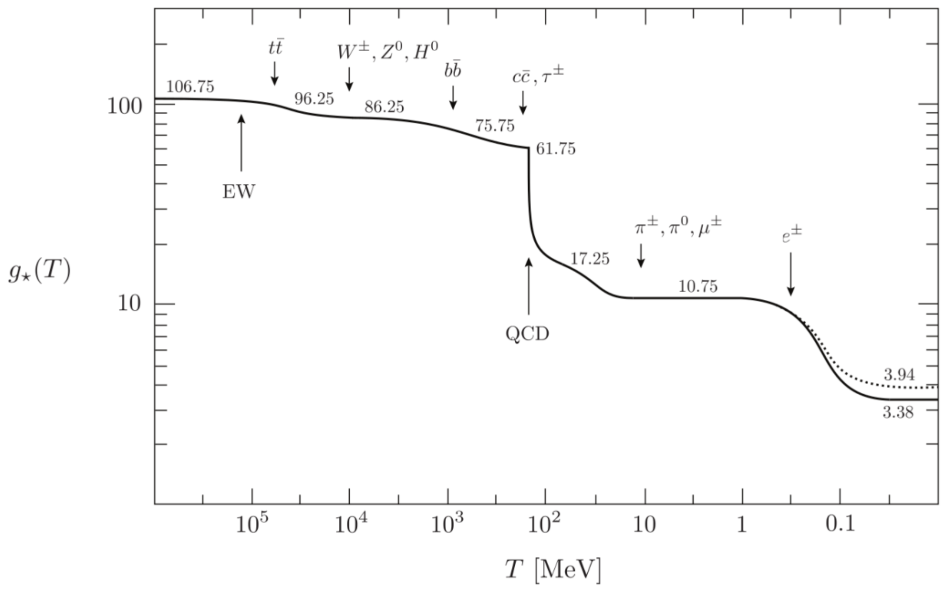
"effective number of neutrino species"

Calculation so far: $N_{\text{eff}} = 3$

↳ Assumes instant decoupling

↳ More realistic: Neutrinos benefit slightly from e^+e^- annihilation

↳ Detailed calculation: $N_{\text{eff}} = 3.0440 \quad \Rightarrow \quad \begin{matrix} g_* = 3.38 \\ g_{*,s} = 3.94 \end{matrix}$



Hubble rate during RD: $H = 1.66\sqrt{g_*}\dfrac{T^2}{M_{\text{pl}}} \rightarrow$ highly sensitive to contribution from νs

7.3 Dark radiation

Consider particle N that decouples form thermal bath at $T_{N,\text{dec}}$

$$\Rightarrow \quad \text{Hot relic for } m_N \ll T_{N,\text{dec}}$$

$$T_{N,\text{eff}} = \left(\frac{g_*}{g_{*,\text{dec}}}\right)^{1/3} T_\gamma$$

$$\rho_N = \xi \frac{g_N}{2}\left(\frac{g_*}{g_{*,\text{dec}}}\right)^{4/3} \rho_\gamma \qquad \text{with } \xi = \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}$$

Contribution to $N_{\text{eff}} = \underbrace{N_{\text{eff, SM}}}_{3.0440} + \Delta N_{\text{eff}}$

$$\Delta N_{\text{eff}} = \frac{\rho_N}{\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\rho_\gamma} = \underbrace{\frac{4}{7}\,\xi\, g_N \left(\frac{11}{4}\cdot\frac{g_*}{g_{*,\text{dec}}}\right)^{4/3}}_{\text{(before neutrino decoupling)}}$$

Consider a Majorana fermion (right-handed neutrino)
with $\xi = \frac{7}{8}$, $g_N = 2$, $T_{\text{dec}} > 100$ GeV ($g_{*,\text{dec}} \approx 100$)

$\Rightarrow \Delta N_{\text{eff}} \approx 0.05$
$\downarrow$
Important target for cosmology

8 Big Bang Nucleosynthesis (BBN)

→ Formation of bound nuclei from protons and neutrons

→ Happens shortly after ν decoupling
$50\,\mathrm{keV} \lesssim T \lesssim 1\,\mathrm{MeV} \overset{\mathrm{RD}}{\Longrightarrow} 1\,\mathrm{s} \lesssim t \lesssim 400\,\mathrm{s}$

→ Earliest time probed by observations

8.1 Qualitative picture

– At $T > 1\,\mathrm{MeV}$ reactions like $\mathrm{p} + \mathrm{e} \longleftrightarrow \mathrm{n} + \nu_e$ are in equilibrium

– At $T \approx 1\,\mathrm{MeV}$, neutrons freeze out, so the only relevant process is $\mathrm{n} \to \mathrm{p} + \mathrm{e} + \bar{\nu}_e$

– At $T \lesssim 0.1\,\mathrm{MeV}$ it becomes favourable to form bound states such as $\mathrm{n}+\mathrm{p} \leftrightarrow \mathrm{D}+\gamma$ (binding energy: $B_0 = 2.2\,\mathrm{MeV}$)

– Only light elements (H, He, Li, Be) can be formed
⇒ Almost all neutrinos end up in ^{4}He ($B_{^4\mathrm{He}} = 28.3\,\mathrm{MeV}$)

8.2 Step 1: Neutron freeze-out

Equilibrium at $T \sim 1\,\mathrm{MeV}$:

$$n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{\frac{3}{2}} e^{(\mu_A - m_A)/T} = e^{\mu_A/T} n_A^{\mu=0} \qquad \text{where } A = \mathrm{p}, \mathrm{n}$$

$e^{\pm}, \nu, \bar{\nu}$ relativistic ⇒ μ negligible

$$\Rightarrow \mu_\mathrm{p} = \mu_\mathrm{n}$$

$$\Rightarrow \frac{n_\mathrm{n}}{n_\mathrm{p}} = \left(\frac{m_\mathrm{n}}{m_\mathrm{p}}\right)^{\frac{3}{2}} e^{-\Delta m/T} \approx e^{-\Delta m/T} \qquad \text{with } \Delta m = m_\mathrm{n} - m_\mathrm{p} = 1.3\,\mathrm{MeV}$$

Neutrons freeze out when $\Gamma_{\mathrm{p}\leftrightarrow\mathrm{n}} < H$

Dimensional analysis: $$H \sim \frac{T^2}{M_\mathrm{pl}}, \quad \Gamma \sim G_F^2 T^5$$

$$\Rightarrow T_\mathrm{n} \sim \left(M_\mathrm{pl} \cdot G_F^2\right)^{-\frac{1}{3}} = 0.8\,\mathrm{MeV}$$

Full calculation: $$T_\mathrm{n} = 0.75\,\mathrm{MeV}$$

$$\Rightarrow \frac{n_\mathrm{n}}{n_\mathrm{p}}(T = T_\mathrm{n}) = 0.18$$

Comment: $T_\mathrm{n} \approx \Delta m$ great coincidence!

↳ Present universe would look very different for $T_\mathrm{n} \gg \Delta m$ or $T_\mathrm{n} \ll \Delta m$

8.3 Step 2: Neutron decay

Shortly after neutron freeze-out: $\mathrm{e}^+\mathrm{e}^-$ annihilation

$\Rightarrow$ $g_s{}^* = 3.94 =$ const

$$\Rightarrow \quad \eta_B = \frac{n_\mathrm{p} + n_\mathrm{n}}{n_\gamma} \sim \frac{n_B}{3} = \text{ const}$$

(baryon-photon ratio, typically $\eta_B \sim 10^{-10}$)

But $X_\mathrm{n} = \dfrac{n_\mathrm{n}}{n_\mathrm{p} + n_\mathrm{n}}$ changes because of neutron decays:

$$X_\mathrm{n}(T) \overset{T \leq T_\mathrm{n}}{=} e^{-t/\tau_\mathrm{n}} X_\mathrm{n}(T_\mathrm{n}) \qquad \text{with } \tau_\mathrm{n} = 880\,\mathrm{s}$$

8.4 Step 3: Deuterium bottleneck

Direct production of ^{4}He from $2\mathrm{p} + 2\mathrm{n}$ strongly suppressed

$\Rightarrow$ Need to produce D first

Consider reaction $\mathrm{p} + \mathrm{n} \longleftrightarrow \mathrm{D} + \gamma \Rightarrow \mu_\mathrm{p} + \mu_\mathrm{n} = \mu_\mathrm{D} \quad (\mu_\mathrm{p} = 0)$

$$\frac{n_\mathrm{D}}{n_\mathrm{p} \cdot n_\mathrm{n}} = \underbrace{\frac{e^{\mu_\mathrm{D}}}{e^{\mu_\mathrm{p}} e^{\mu_\mathrm{n}}}}_{=1} \frac{n_\mathrm{D}^{\mu=0}}{n_\mathrm{p}^{\mu=0} n_\mathrm{n}^{\mu=0}} = \frac{g_\mathrm{D}}{g_\mathrm{p} g_\mathrm{n}} \left(\frac{2\pi m_\mathrm{D}}{m_\mathrm{n} m_\mathrm{p} T} \right)^{\frac{3}{2}} e^{\frac{m_\mathrm{n} + m_\mathrm{p} - m_\mathrm{D}}{T}}$$

Using $g_\mathrm{D} = 3, \quad g_\mathrm{p} = g_\mathrm{n} = 2, \quad m_\mathrm{n} \approx m_\mathrm{p} \approx \dfrac{m_\mathrm{D}}{2}, \quad m_\mathrm{n} + m_\mathrm{p} - m_\mathrm{D} = B_\mathrm{D}$

$$\frac{n_\mathrm{D}}{n_\mathrm{p} \cdot n_\mathrm{n}} = \frac{3}{4} \left(\frac{4\pi}{m_\mathrm{p} T} \right)^{\frac{3}{2}} e^{B_\mathrm{D}/T}$$

Using $X_\mathrm{n}\,(1 - X_\mathrm{n}) = \dfrac{n_\mathrm{p} \cdot n_\mathrm{n}}{n_B^2}$ and $n_B = \eta_B n_\gamma$

$$\Rightarrow \quad \frac{n_\mathrm{D}}{n_B} = \frac{3}{4} X_\mathrm{n}\,(1 - X_\mathrm{n})\,\eta_B n_\gamma \left(\frac{4\pi}{m_\mathrm{p} T} \right)^{\frac{3}{2}} e^{B_\mathrm{D}/T} \sim 10^{-10} \left(\frac{T}{m_\mathrm{p}} \right)^{\frac{3}{2}} e^{B_\mathrm{D}/T}$$

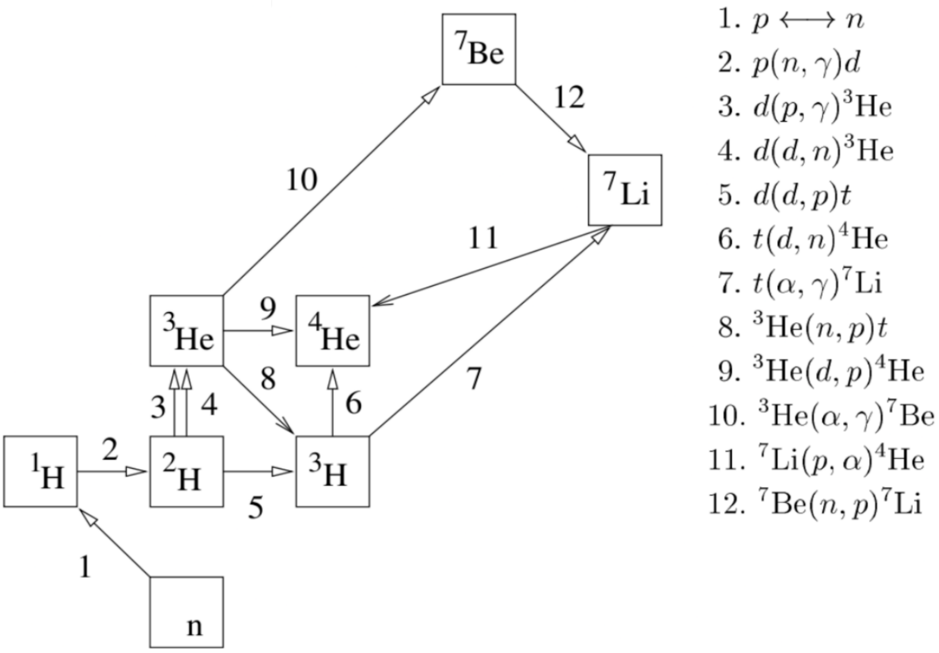
For $T \gtrsim 100\,\text{keV}$: $\quad \dfrac{n_\text{D}}{n_B} \ll 1$

D production efficient $\Leftrightarrow \dfrac{n_\text{D}}{n_B} = 1$

$$\Rightarrow \frac{B_\text{D}}{T} = -\ln\left[10^{-10}\left(\frac{T}{m_\text{p}}\right)^{\frac{3}{2}}\right] \sim 30 \Rightarrow T_\text{D} \approx 80\,\text{keV} \Rightarrow t_\text{D} \approx 150\,\text{s}$$

8.5 Step 4: Deuterium burning

For $T < T_\text{D}$: $\quad$ Chain of nuclear reactions



Primordial nuclear reactions

$$\left.\begin{array}{lcl} \text{D}+\text{D} & \longrightarrow & {}^3\text{H}+\text{p} \\ \text{D}+\text{D} & \longrightarrow & {}^3\text{He}+\text{n} \\ \text{D}+\text{n} & \longrightarrow & {}^3\text{H}+\gamma \\ \text{D}+\text{p} & \longrightarrow & {}^3\text{He}+\gamma \end{array}\right\} \Longrightarrow \begin{array}{lcl} \text{D}+{}^3\text{He} & \longrightarrow & {}^4\text{He}+\text{p} \\ \text{D}+{}^3\text{H} & \longrightarrow & {}^4\text{He}+\text{n} \end{array}$$

$\Rightarrow$ Almost all neutrons end up in ^{4}He

$$\Rightarrow \frac{n_{^4\text{He}}}{n_B} = \frac{1}{2}X_\text{n}\left(T = T_\text{D}\right) = \frac{1}{2}e^{-t_\text{D}/\tau_\text{n}}X_\text{n}\left(T = T_\text{n}\right) = 0.063 \approx \frac{1}{16}$$

$$Y_\text{p} = \frac{\rho_{^4\text{He}}}{\rho_B} \approx 4\,\frac{n_{^4\text{He}}}{n_B} = 0.25$$

Fraction of ^{4}He remains constant until star formation starts, which produces heavier elements (e.g. $^4\text{He} + {}^4\text{He} + {}^4\text{He} \to {}^{12}\text{C}$)

In some regions with low star formation rates $Y_\text{p}(\text{today}) \approx Y_\text{p}(\text{BBN})$
$\Rightarrow$ Possible to directly measure Y_p

Result: $Y_\text{p} = 0.245 \pm 0.003 \to$ Spectacular success!

8.6 Determining η_B

Y_p depends on η_B only logarithmically through $T_\text{D} \longrightarrow$ insufficient for measuring η_B

Instead: Consider end of D burning

$$\Gamma_\text{D} = n_\text{D} \cdot \langle \sigma v \rangle_\text{D} < H$$

↳ Complicated nuclear physics → can be measured

Freeze-out happens for $T_\text{fo} \approx 65$ keV

$$\Rightarrow n_D\left(T_\text{fo}\right) = \frac{H\left(T_\text{fo}\right)}{\langle \sigma v \rangle_\text{D}\left(T_\text{fo}\right)} \approx 10^{14}\ \text{cm}^{-3}$$

Using $n_\text{p} = n_B - 4 n_{^4\text{He}} \approx \frac{3}{4} n_B = \frac{3}{4} \eta_B n_\gamma$

$$\frac{\text{D}}{\text{H}} = \frac{n_\text{D}}{n_\text{p}} = \frac{1}{\eta_B} \cdot \frac{4}{3} \cdot \frac{n_\text{D}\left(T_\text{fo}\right)}{n_\gamma\left(T_\text{fo}\right)} = \frac{1}{\eta_B} \cdot 1.6 \cdot 10^{-14}$$

Observations: $\frac{\text{D}}{\text{H}} = (2.55 \pm 0.03) \cdot 10^{-5}$

$\Rightarrow \eta_B = (6.2 \pm 0.2) \cdot 10^{-10}$

η_B remains constant until today $\Rightarrow \Omega_B h^2 = \frac{m_\text{p} \cdot \eta_B \cdot n_{\gamma,0}}{\rho_c / h^2} = 0.022 \pm 0.001$

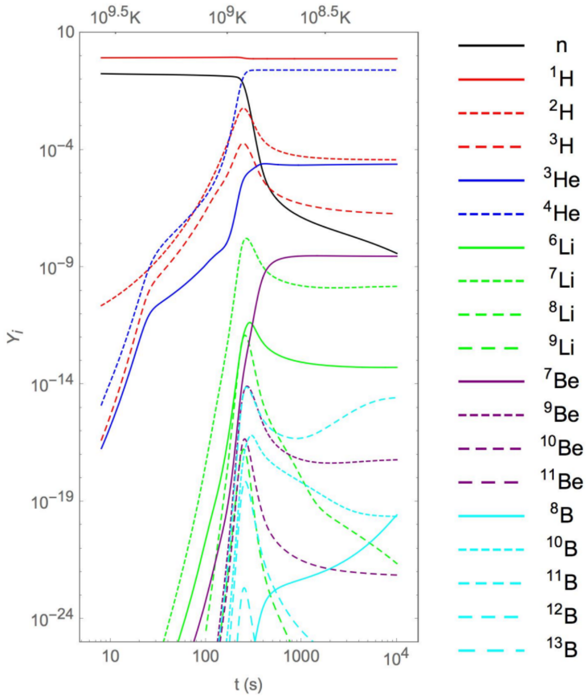
↳ Baryons only constitute $\sim 4\%$ of the energy density of the present universe!

Comment: Also possible to predict $\frac{^7\text{Li}}{\text{H}}$

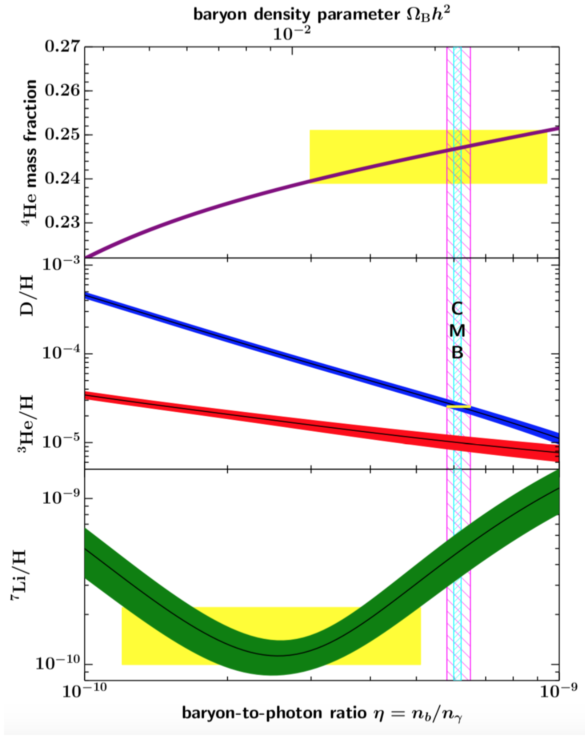
↳ Inferred value of η_B too small

↳ Theory uncertainty? Measurement error? New physics?

Note: BBN powerful probe of physics beyond Standard Model



(a) Dependence on t



(b) Dependence on η_B

Example: Y_p very sensitive to Hubble rate at $T \sim 1\,\mathrm{MeV}$

$\Rightarrow$ Confirms SM prediction $N_\mathrm{eff} = 3.0440$

$\Rightarrow$ Upper bound: $\Delta N_\mathrm{eff} < 0.4\,(95\%$ C.L.)

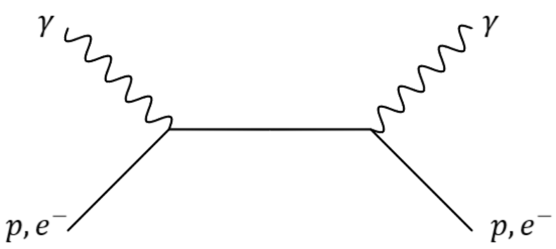
$\Rightarrow$ Excludes additional light particles in thermal equilibrium

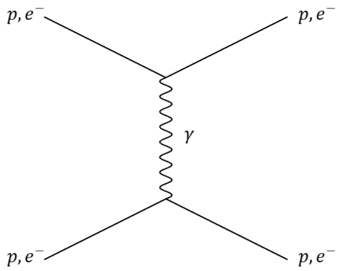
9 Recombination

↳ Formation of neutral hydrogen (neglect He for today)

Universe at $T = 1\,\text{eV}$:

- $\text{p}, \text{e}^-, \gamma$ tightly coupled:





- Equilibrium between free particles and bound H:

$$\text{e}^- + \text{p} \longleftrightarrow \text{H} + \gamma$$

$$n_i^{\text{eq}} = g_i \left(\frac{m_i T}{2\pi}\right)^{\frac{3}{2}} e^{(\mu_i - m_i)/T} \qquad \text{with } \mu_\text{p} + \mu_e = \mu_\text{H}$$

$$\Rightarrow \underbrace{\left(\frac{n_\text{H}}{n_e \cdot n_\text{p}}\right)_{\text{eq}}}_{n_\text{H}/n_e^2} = \underbrace{\frac{g_\text{H}}{g_e \cdot g_\text{p}}}_{=\frac{4}{2\cdot 2}} \left(\underbrace{\frac{m_\text{H}}{m_e m_\text{p}}}_{\approx 1/m_e} \frac{2\pi}{T}\right)^{\frac{3}{2}} \underbrace{e^{(m_\text{p} + m_e - m_\text{H})/T}}_{\substack{\exp(B_\text{H}/T) \\ \text{with } B_\text{H} = 13.6 \text{ eV}}} \qquad (n_e = n_\text{p} \text{ from charge neutrality})$$

$$\Rightarrow \left(\frac{n_\text{H}}{n_e^2}\right)_{\text{eq}} = \left(\frac{2\pi}{m_e T}\right)^{\frac{3}{2}} e^{B_\text{H}/T}$$

Convenient to define

$$X_e = \frac{n_e}{n_\text{p} + n_\text{H}} \qquad \text{(free electron fraction)}$$

Note:

$$n_\text{p} + n_\text{H} = \underset{\downarrow}{0.75} \cdot \eta_B \cdot n_\gamma \qquad \left(n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3\right)$$

approximate fraction of H (rest is He)

$$\Rightarrow 1 - X_e = \frac{\cancel{n_\text{p}} + n_\text{H} - \cancel{n_e}}{n_\text{p} + n_\text{H}}$$

$$\Rightarrow \frac{1 - X_e}{X_e^2} = \frac{n_\text{H}}{n_e^2}(n_\text{p} + n_\text{H})$$

$$\Rightarrow \boxed{\left(\frac{1-X_e}{X_e^2}\right)_{\text{eq}} = \frac{2\zeta(3)}{\pi^2}\left(\frac{2\pi T}{m_e}\right)^{\frac{3}{2}} e^{B_{\text{H}}/T}\cdot 0.75\cdot \eta_b} \qquad \text{(Saha equation)}$$

For $T \gtrsim B_{\text{H}}$: rhs tiny (remember: $\eta_B \sim 10^{-10}$) $\Rightarrow X_e \approx 1$

For $T \ll B_{\text{H}}$: rhs increases $\Rightarrow X_e$ decreases

Recombination happens when

$$\begin{aligned}\frac{B_{\text{H}}}{T} &\approx -\log\left[\frac{2\zeta(3)}{\pi^2}0.75\eta_b\left(\frac{2\pi T}{m_e}\right)^{\frac{3}{2}}\right]\\ &= \underbrace{-\log\left[\frac{2\zeta(3)}{\pi^2}0.75\eta_b\left(\frac{2\pi B_{\text{H}}}{m_e}\right)^{\frac{3}{2}}\right]}_{=35.9} + \underbrace{\frac{3}{2}\log\frac{B_{\text{H}}}{T}}_{\text{can be neglected}}\end{aligned}$$

$$\Rightarrow T_{\text{rec}} \approx 0.38\,\text{eV}$$

$$T_{\text{rec}} = T_0\,(1+z_{\text{rec}}) \Rightarrow z_{\text{rec}} \approx 1600$$

Numerical solution: $T = 0.4\,\text{eV}: \quad X_e = 0.999$
$T = 0.3\,\text{eV}: \quad X_e = 0.15$

$T_{\text{rec}} \ll B_{\text{H}}$ because $n_\gamma \gg n_e, n_{\text{p}}$

$\Rightarrow$ Enough high-energy photons in tail of distribution to keep H ionized

$\Rightarrow$ Recombination happened after M-R equality

$$\Rightarrow \quad t_{\text{rec}} = \frac{2}{3}H(T_{\text{rec}})^{-1} = \frac{2}{3H_0\sqrt{\Omega_{\text{M}}\,(1+z_{\text{rec}})^3}} \approx 2.6\cdot 10^5\,\text{years}$$

9.1 Photon decoupling

As long as $X_e \approx 1$, photons experience frequent interactions:

$$\text{e}^- + \gamma \longleftrightarrow \text{e}^- + \gamma$$

$$\sigma_{\text{T}} = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} \approx 0.67\cdot 10^{-24}\,\text{cm}^2 \qquad \text{(Thomson cross section)}$$

$$\Gamma_\gamma = n_e\sigma_{\text{T}} = X_e\sigma_{\text{T}}(n_{\text{p}}+n_{\text{H}})$$

With decreasing X_e, Γ_γ decreases as well

$\Rightarrow$ Photons decouple for $\Gamma_\gamma(T_{\text{dec}}) = H(T_{\text{dec}})$

$$\text{MD:} \quad H\left(T_{\text{dec}}\right) = H_0\sqrt{\Omega_{\text{M}}}\left(\frac{T_{\text{dec}}}{T_0}\right)^{\frac{3}{2}}$$

$$\Rightarrow \quad \underset{\downarrow}{X_e}\left(T_{\text{dec}}\right)T_{\text{dec}}^{3/2} = \frac{\pi^2}{2\zeta(3)}\frac{H_0\sqrt{\Omega_{\text{M}}}}{0.75\eta_B\sigma_{\text{T}}T_0^{3/2}}$$

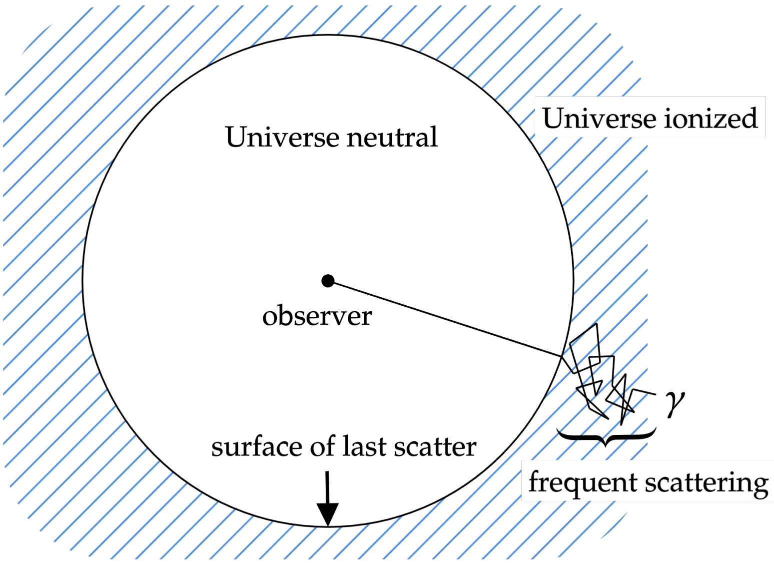
Can be estimated from Saha eq.

$$\Rightarrow \quad T_{\text{dec}} \approx 0.27\,\text{eV} \Rightarrow z_{\text{dec}} \approx 1100 \Rightarrow t_{\text{dec}} \approx 380000\,\text{yrs}$$

Many refinements needed:

- Electrons not in perfect equilibrium at T_{dec}
  $\Rightarrow$ Need to solve Boltzmann equation to get X_e
- Need to include excited hydrogen $(2s, 2p)$ in calculation

But final result robust, because X_e drops exponentially $\Rightarrow$ photon decoupling very sudden



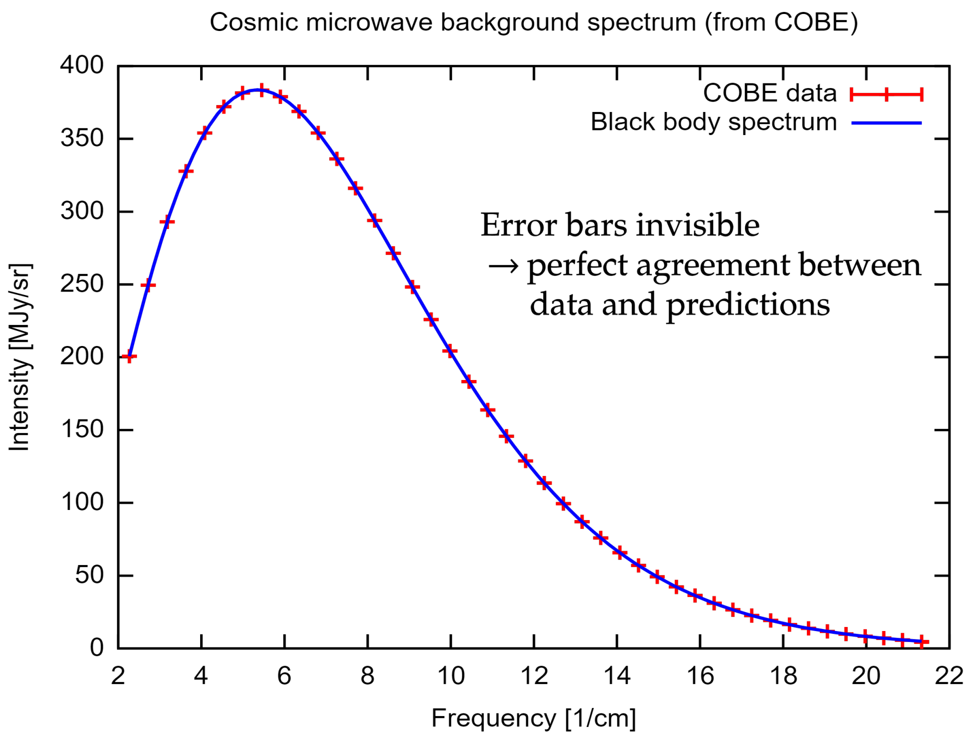
We see the surface of last scatter in every direction at a distance of $\sim$ 13.5 Glyr

Beyond it, the Universe is intransparent

The photons emitted from this surface form the CMB

9.2 Cosmic Microwave Background

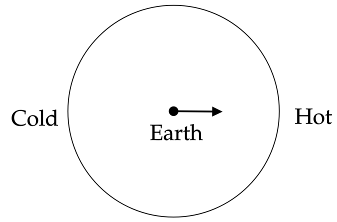
↳ First discovered in 1965 by accident



$T_{\text{CMB}} = 2.7255 \pm 0.0006\,\text{K}$

When seen from the Earth, T_{CMB} is not the same in every direction

↳ Doppler effect due to relative motion between Earth and CMB

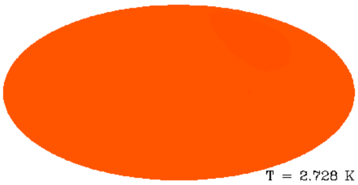


$$p_{\text{obs}}(\underbrace{\vec{n}}_{\substack{\text{unit}\\ \text{vector}}}) = \frac{p}{\gamma(1-\vec{n}\cdot\vec{v})} \overset{v\ll 1}{\approx} p(1+\vec{n}\cdot\vec{v})$$

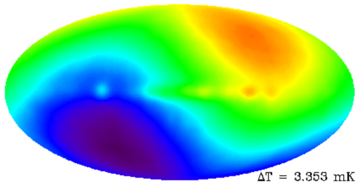
$$\Rightarrow \quad \frac{\delta T(\vec{n})}{T} = \frac{p_{\text{obs}}(\vec{n})-p}{p} = \vec{n}\cdot\vec{v} \quad \Rightarrow \text{ dipole anisotropy}$$

Fit to data gives $v = 368$ km/s

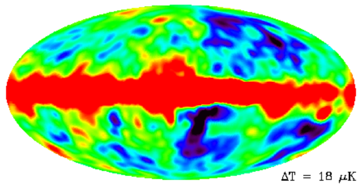
→ Subtract dipole to reveal primordial anisotropy



(a) CMB monopole

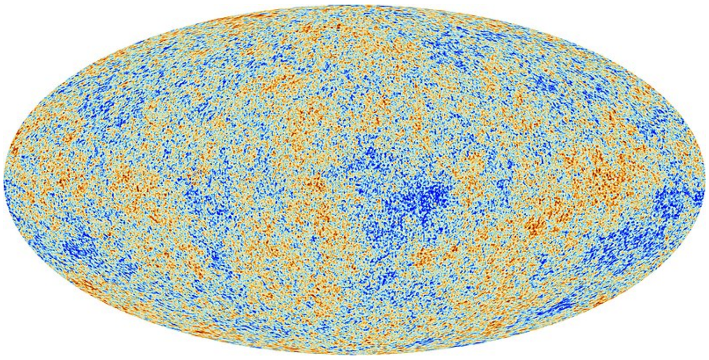


(b) CMB dipole

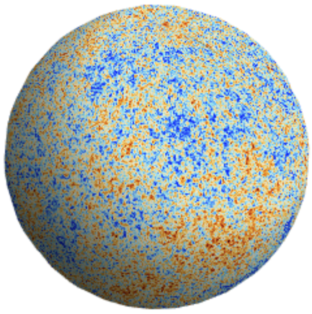


(c) Primordial anisotropy

Removing galactic backgrounds:



(a) Mollweide projection



(b) without projection

Conclusion: Universe is <u>not</u> perfectly homogeneous at $T = T_{\text{dec}}$
$\Rightarrow$ need to study perturbations

Summary: