§2 Brief Introduction to General Relativity

Recap: Special relativity

Define $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad \text{(summation convention)}$$

with $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ and $\mu, \nu = 0, 1, 2, 3$

$\Rightarrow$ ds^2 is invariant under Lorentz transformation

$$y^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow dy^\mu = \frac{\partial y^\mu}{\partial x^\nu} dx^\nu = \Lambda^\mu{}_\nu dx^\nu$$

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$$

Quantities that transform like dx^μ are called "contravariant vectors"

Example: Consider world line $X^\mu(\tau)$ of particle

$$\Rightarrow U^\mu = \frac{dX^\mu}{d\tau} \quad \text{is contravariant vector}$$

Partial derivatives transform in the opposite way:

$$\frac{\partial x^\nu}{\partial x^\mu} = \frac{\partial y^\nu}{\partial y^\mu} = \delta^\nu_\mu = \begin{cases} 1 & \mu = \nu \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \frac{\partial}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu}$$

Any quantity transforming like $\frac{\partial}{\partial x^\mu}$ is called "covariant vector"

Generalisation to tensors:

$$T^{\mu}{}_{\nu}(y) = \frac{\partial y^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial y^\nu} T^\rho{}_\sigma(x) = \Lambda^\mu{}_\rho (\Lambda^{-1})^\sigma{}_\nu T^\rho{}_\sigma(x)$$

$\Rightarrow$ $\eta_{\mu\nu}$ is a rank-2 covariant tensor

Convenient to define inverse metric $\eta^{\mu\nu}$:

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho = \mathrm{diag}(1,1,1,1)$$

↳ Can be used to "pull" indices up and down:

$$T^{\mu\nu} = \eta^{\mu\rho} T^\nu{}_\rho \quad ; \quad T_{\mu\nu} = \eta_{\mu\rho} T^\rho{}_\nu$$

2.1 Non-inertial reference frames

Lorentz transformations do not introduce ficticious forces. Consider instead general transformation

$$y^\mu = y^\mu(x^\nu)$$

$$\Rightarrow \quad ds^2 = \eta_{\mu\nu}\, dy^\mu dy^\nu = \left(\eta_{\mu\nu} \frac{\partial y^\mu}{\partial x^\rho} \frac{\partial y^\nu}{\partial x^\sigma}\right) dx^\rho dx^\sigma$$

$$\equiv \underbrace{g_{\rho\sigma}}_{\text{may depend on } x}\, dx^\rho dx^\sigma$$

Consider motion of inertial particle $Y^\mu(\tau)$:

$$\frac{d^2 Y}{d\tau^2} = 0 \quad \text{(no acceleration)}$$

New coordinate system:

$$\frac{d^2 y^\mu}{d\tau^2} = \frac{d}{d\tau}\frac{dy^\mu}{d\tau} = \frac{d}{d\tau}\left(\frac{\partial y^\mu}{\partial x^\nu}\overbrace{\frac{dx^\nu}{d\tau}}^{=U^\nu}\right)$$

$$= U^\nu\left(\frac{d}{d\tau}\frac{\partial y^\mu}{\partial x^\nu}\right) + \frac{\partial y^\mu}{\partial x^\nu}\frac{dU^\nu}{d\tau} = 0$$

Using $\frac{d}{d\tau}\frac{\partial y^\mu}{\partial x^\nu} = U^\rho \frac{\partial^2 y^\mu}{\partial x^\rho \partial x^\nu}$ and $\frac{\partial y^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial y^\rho} = \delta^\mu_\rho$

$$\Rightarrow \underbrace{\frac{dU^\rho}{d\tau} + U^\mu U^\nu \underbrace{\left(\frac{\partial^2 y^\sigma}{\partial x^\mu \partial x^\nu}\frac{\partial x^\rho}{\partial y^\sigma}\right)}_{\equiv \Gamma^\rho_{\mu\nu}}}_{\neq 0 \text{ in general}} = 0$$

$\rightarrow$ non-inertial frame

Using
$$\frac{\partial g_{\rho\sigma}}{\partial x^\lambda} = \eta_{\mu\nu}\frac{\partial}{\partial x^\lambda}\left(\frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma}\right)$$

$$= \eta_{\mu\nu}\left(\frac{\partial^2 y^\mu}{\partial x^\lambda \partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma} + \frac{\partial^2 y^\nu}{\partial x^\lambda \partial x^\sigma}\frac{\partial y^\mu}{\partial x^\rho}\right)$$

$\eta_{\mu\nu}\frac{\partial y^\nu}{\partial x^\sigma} = g_{\kappa\sigma}\frac{\partial x^\kappa}{\partial y^\mu}$

$$= g_{\kappa\sigma}\Gamma^\kappa_{\lambda\rho} + g_{\kappa\rho}\Gamma^\kappa_{\lambda\sigma}$$

we obtain

$$\boxed{\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)}$$

"Christoffel symbols"

Convenient to define covariant derivative:

Scalars: $\nabla_\mu X = \partial_\mu X$

Vector: $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\rho} X^\rho$

$\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^\rho_{\mu\nu} X_\rho$

$\Rightarrow \nabla_\mu (X^\nu X_\nu) = \partial_\mu (X^\nu X_\nu)$

Tensor: $\nabla_\mu T^\nu{}_\rho = \partial_\mu T^\nu{}_\rho + \Gamma^\nu_{\mu\sigma} T^\sigma{}_\rho - \Gamma^\sigma_{\mu\rho} T^\nu{}_\sigma$

$\Rightarrow \nabla_\mu g_{\rho\sigma} = 0$

$$\Rightarrow \frac{dU^\rho}{d\tau} + \Gamma^\rho_{\mu\nu} U^\mu U^\nu = \frac{\partial U^\rho}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \Gamma^\rho_{\mu\nu} U^\mu U^\nu$$

$$= U^\mu \left(\partial_\mu U^\rho + \Gamma^\rho_{\mu\nu} U^\nu\right)$$

$$= U^\mu \nabla_\mu U^\rho = 0$$

"geodesic equation"

Note: Can also write geodesic eq. in terms of

$P^\mu = m U^\mu \Rightarrow P^\mu \nabla_\mu P^\rho = 0$

$\hookrightarrow$ valid also for massless particles

2.2 Curved spacetime

Metric $g_{\mu\nu}$ can describe not only non-inertial frames but also general curved spacetime

In such a space time, covariant derivatives do not commute:

$$\nabla_\mu \nabla_\nu A^\sigma - \nabla_\nu \nabla_\mu A^\sigma = R^\sigma{}_{\mu\nu\rho} A^\rho$$

with $$\boxed{R^\sigma{}_{\mu\nu\rho} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\rho \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\lambda\nu} \Gamma^\lambda_{\mu\rho} - \Gamma^\sigma_{\lambda\rho} \Gamma^\lambda_{\mu\nu}}$$

"Riemann tensor"

Interpretation: Consider two particles with separation B^μ travelling with the same velocity U^μ

$$\frac{D^2 B^\mu}{D\tau^2} = - R^\mu{}_{\nu\rho\sigma} U^\nu U^\sigma B^\rho$$

$\uparrow \; \frac{D}{D\tau} = U^\nu \nabla_\nu$ $\qquad \neq 0$ in curved spacetime

Convenient to define

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \qquad \text{"Ricci tensor"}$$

$$R = g^{\mu\nu} R_{\mu\nu} \qquad \text{"Ricci scalar"}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad \text{"Einstein tensor"}$$

Comment: Can show that $\nabla^\mu G_{\mu\nu} = 0$

2.3 Equivalence principle

Gravity is locally indistinguishable from acceleration (i.e. coordinate transformation to non-inertial frame)

$\Rightarrow$ Effect of gravity fully captured by metric $g_{\mu\nu}$

How does metric depend on gravitating matter?

Consider a perfect fluid with density ρ and pressure p assumed to be homogeneous and isotropic in its rest frame $(U^\mu = (1,0,0,0))$

Define energy-momentum tensor

$$\boxed{T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)}$$

in rest frame. In general frame with velocity U^μ

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu - p\, g_{\mu\nu}$$

$$= \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{momentum density} & \text{stress tensor} \end{array}\right)$$

E-p conservation imply

$$\nabla^\mu T_{\mu\nu} = 0 \qquad \text{for all } \nu$$

Both $G_{\mu\nu}$ and $T_{\mu\nu}$ are covariantly conserved

Tempting to write $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$

unknown

To determine κ^2 consider metric

$$ds^2 = c^2 dt^2 \left(1 + \frac{2\Phi(\vec{x})}{c^2}\right) - dx^2 - dy^2 - dz^2$$

$\Phi/c^2 \ll 1$

Find $\Gamma^{i}_{00} = \underbrace{\frac{1}{2} g^{i\sigma}}_{= -\delta^{i\sigma}} \left(\underbrace{\frac{\partial g_{0\sigma}}{\partial x^0} + \frac{\partial g_{0\sigma}}{\partial x^0}}_{=0 \text{ for } \sigma = i} - \frac{\partial g_{00}}{\partial x^\sigma} \right)$ ($i = 1,2,3$)

$$= \frac{1}{2} \frac{\partial}{\partial x^i} g_{00} = \frac{1}{c^2} \frac{\partial \Phi}{\partial x^i}$$

=> Geodesic eq. for non-rel. particle

$$\ddot{x}^i = -\Gamma^{i}_{00} u^0 u^0 = -\frac{\partial \Phi}{\partial x^i}$$

↳ Φ acts like Newtonian potential

Now calculate $g^{\mu\nu} G_{\mu\nu} = -R = 2\nabla^2\Phi$

$$g^{\mu\nu} T_{\mu\nu} = \rho - 3p \approx \rho$$

non-rel.

$$\Rightarrow 2\nabla^2 \Phi = \kappa^2 \rho$$

Compare to Poisson eq. $\nabla^2 \Phi = 4\pi G \rho$

Newton's constant

$\Rightarrow \kappa^2 = 8\pi G$

$$\Rightarrow \boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}}$$ Einstein equation