§3 The FLRW metric

Recap: Geometry of space-time described by metric $g_{\mu\nu}$

↳ 10 indep. functions of $(t, \vec{x})$

Important simplification: Universe observed to be homogeneous & isotropic

↳ $g_{\mu\nu}$ indep. of $\vec{x}$

↳ $g_{\mu\nu}$ invariant under rotation

Example: Static flat space

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

$$= dt^2 - dr^2 - r^2 \underbrace{(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2)}_{= d\Omega^2}$$

More general: Allow expansion/contraction of spatial part with time

$$ds^2 = dt^2 - a(t)^2 \left[dr^2 + r^2 d\Omega^2 \right]$$

↑ "scale factor"

$a(t)$ is dimensionless ⇒ only ratio $\frac{a(t_1)}{a(t_2)}$ meaningful

Define $\boxed{H(t) = \frac{\dot{a}(t)}{a(t)}}$ (Hubble rate)

Most general: Allow const. spatial curvature

$$\boxed{\begin{aligned} ds^2 &= dt^2 - a(t)^2\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right] \\ \text{with} \quad \kappa &= \begin{cases} +1 & \text{pos. curvature} \quad (3\text{-sphere}) \\ 0 & \text{flat} \quad (3\text{-plane}) \\ -1 & \text{neg. curvature} \quad (3\text{-hyperboloid}) \end{cases} \end{aligned}}$$

"Friedmann-Lemaître-Robertson-Walker (FLRW) metric"

Notes: - For $\kappa \neq 0$, r must be dim.less and $a(t)$ has dim. of length

↳ Interpretation: $a(t)$ = radius of curvature

- Sometimes convenient to define

$$d\chi \equiv \frac{dr}{\sqrt{1-\kappa r^2}} \qquad d\eta = \frac{dt}{a(t)}$$

↳ "coordinate distance" ↳ "conformal time"

$$\Rightarrow \quad ds^2 = a(\eta)^2\left[d\eta^2 - \left(d\chi^2 + S_\kappa^2(\chi)\, d\Omega^2\right)\right]$$

$$\text{with} \quad S_\kappa(\chi) \equiv \begin{cases} \sin\chi & \kappa = 1 \\ \chi & \kappa = 0 \\ \sinh\chi & \kappa = -1 \end{cases}$$

Consider particle at rest: $x^\mu = (\tau, x_0, y_0, z_0)$

$u^\mu = (1, 0, 0, 0)$

Geodesic equation:

$$0 = \underbrace{\frac{du^\rho}{d\tau}}_{=0} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = \Gamma^\rho_{00}$$

$$= \frac{1}{2} g^{\rho\sigma} \left(\partial_0 \underbrace{g_{0\sigma}}_{=\delta_{0\sigma}} + \partial_0 \underbrace{g_{\sigma 0}}_{=\delta_{0\sigma}} - \partial_\sigma g_{00} \right)$$

$$= 0$$

$\Rightarrow$ Particles at rest are free (no forces)

<u>But:</u> Physical distance to origin changes with time t:

$$ds^2 = a(t)^2 \frac{dr^2}{1 - kr^2}$$

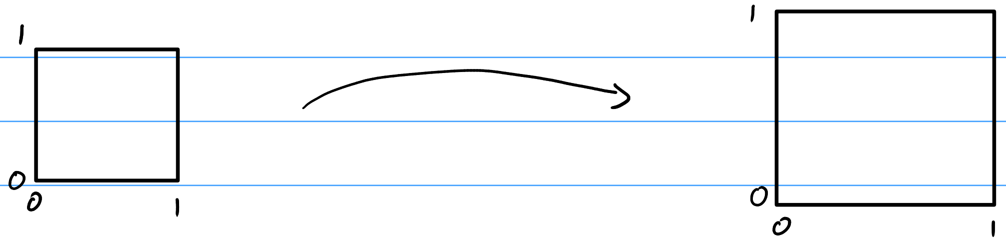
$$\Rightarrow d(r, t) = a(t) \int_0^r \frac{dr}{\sqrt{1 - kr^2}}$$

physical distance

$$= a(t) \begin{cases} \arcsin r & k = 1 \\ r & k = 0 \\ \operatorname{arcsinh} r & k = -1 \end{cases}$$

coordinate distance

$a(t_1)$ $\qquad\qquad$ $a(t_2) > a(t_1)$



x^i : "comoving coordinate"

$X^i = a(t)x^i$: "physical coordinate"

$$V^i = \frac{dX^i}{dt} = \underbrace{a(t)\frac{dx^i}{dt}}_{\text{peculiar velocity}} + \underbrace{HX^i}_{\text{"Hubble flow"}}$$

Now consider particle with momentum P^μ

$$0 = P^\alpha \partial_\alpha P^\mu + \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta$$

For $\mu = 0$

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} \underbrace{g^{0\lambda}}_{=\delta^{0\lambda}} \Big(\partial_\alpha \underbrace{g_{\beta\lambda}}_{=\delta_{\beta 0}} + \partial_\beta \underbrace{g_{\alpha\lambda}}_{=\delta_{\alpha 0}} - \partial_\lambda g_{\alpha\beta} \Big)$$

$$= -\frac{1}{2} \partial_0 g_{\alpha\beta}$$

$$\Rightarrow \Gamma^0_{00} = \Gamma^0_{0i} = 0$$

$$\Gamma^0_{ij} = -\frac{1}{2}\partial_0 g_{ij} = \frac{1}{2}\partial_t a(t)^2 \gamma_{ij}$$

spatial metric

$$= \dot{a}(t)\, a(t)\, \gamma_{ij}$$

$$\Rightarrow 0 = P^\alpha \partial_\alpha P^0 + \dot{a} a \gamma_{ij} P^i P^j$$

Homogeneity of space: $\partial_i P^0 = 0$

Use $P^0 = E$, $\quad -g_{ij} P^i P^j = a^2 \gamma_{ij} P^i P^j = p^2$

physical 3-momentum

$$\Rightarrow E \frac{dE}{dt} = -\frac{\dot{a}}{a} p^2 = -H p^2$$

$$E^2 - p^2 = m^2 \Rightarrow E\,dE = p\,dp \Rightarrow \frac{\dot{p}}{p} = -\frac{\dot{a}}{a}$$

For $m = 0$: $\quad p = E \sim \frac{1}{a}$

↳ Energy of massless particles decreases with increasing scale factor

For $m \neq 0$:

$$P^i = m U^i = m \frac{dx^i}{d\tau} = m \frac{dt}{d\tau} v^i = \frac{m v^i}{\sqrt{1 - v^2}}$$

$$\Rightarrow \frac{m v}{\sqrt{1 - v^2}} \sim \frac{1}{a}$$

↳ Peculiar velocity decreases

↳ Particles converge onto Hubble flow

3.1 Redshift

Photons have $\lambda = \frac{h}{p} \sim a(t)$

Classical interpretation: Expansion of space streches wavelength

Consider photon emitted at time t_i with λ_i

Present universe: $$\lambda_0 = \lambda_i \frac{a_0}{a(t_i)}$$
$$= \lambda_i (1 + z(t_i))$$

with $$\boxed{z(t) = \frac{a_0}{a(t)} - 1}$$ "redshift"

If λ_i is known (e.g. spectral line), we can infer $z(t_i)$ from λ_0

$\Rightarrow$ Infer time since emission

$\Rightarrow$ Infer distance of source

Useful relations:

$$dz = -\frac{a_0}{a^2} da = -\frac{a_0}{a} H \, dt = -(1+z) H \, dt$$

For nearby sources

$$a(t_i) = a_0\left(1 + (t_i - t_0)\,\underset{\left.\frac{\dot{a}}{a}\right|_{t=t_0}}{H_0} + \dots\right)$$

"Hubble constant"

$$\Rightarrow \quad z(t_i) \approx H_0 \underbrace{(t_0 - t_i)}_{\approx d}$$

(distance to emitter)

$$\Rightarrow \quad \boxed{z \overset{z \ll 1}{\approx} H_0\, d}$$

"Hubble's law"

↳ redshift proportional to distance

↳ can be used to measure H_0 (inaccurate)

Note: $H = \frac{\dot{a}}{a} \quad \Rightarrow \quad [H_0] = [\text{time}]^{-1}$

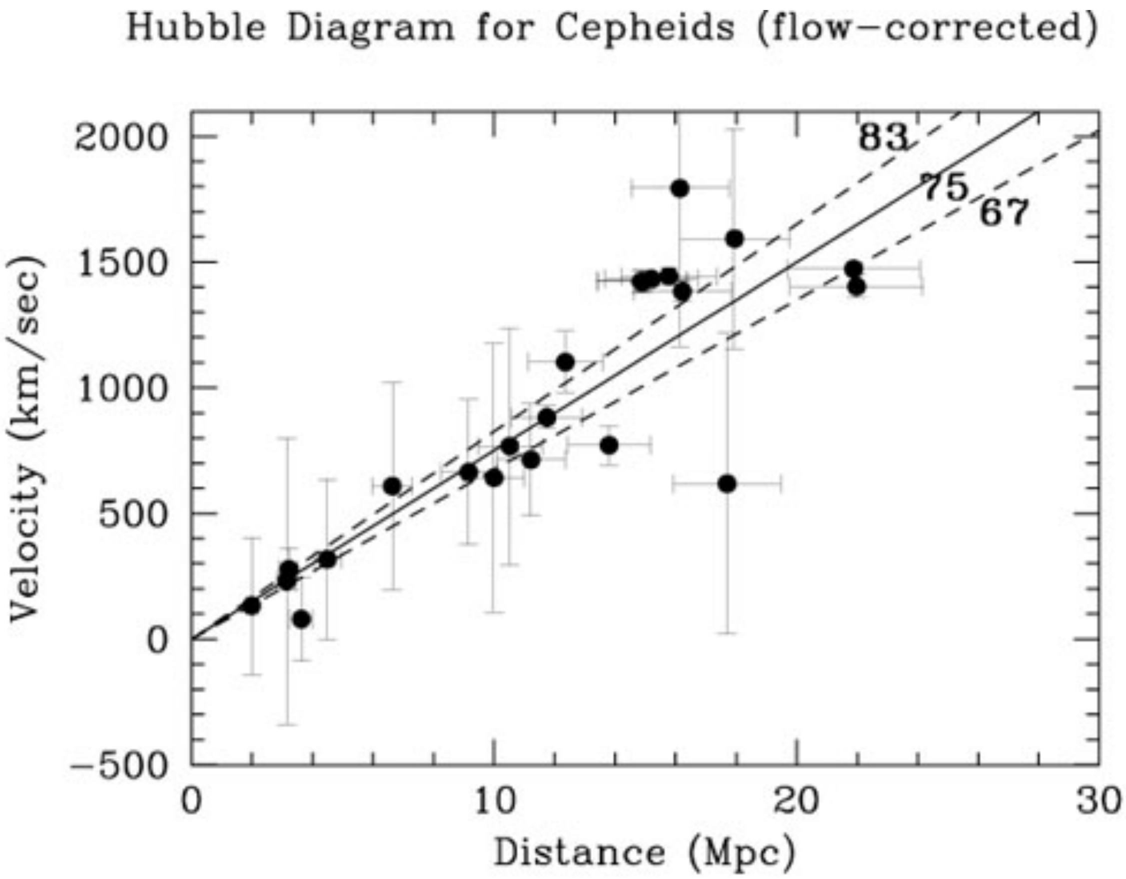
$[d] = [\text{distance}]$

$\Rightarrow \quad [z] = [\text{velocity}]$

Convention: $[H_0] = \text{km s}^{-1}\ \text{Mpc}^{-1}$

$[d] = \text{Mpc}$

$[z] = \text{km s}^{-1}$


Hubble Diagram for Cepheids (flow−corrected)
Velocity (km/sec)
2000
1500
1000
500
0
−500
0
10
20
30
Distance (Mpc)
83
75
67