§4 Dynamics of Cosmological Expansion

Recap: $$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right]$$

$$\Rightarrow g_{00} = 1, \quad g_{ij} = -a(t)^2 \gamma_{ij}$$

What determines $a(t)$?

Einstein eq.: $$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Let's calculate $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$\Gamma^i_{0j} = \frac{1}{2} \underbrace{g^{i\mu}}_{=0 \text{ for } \mu=0} (\partial_0 g_{\mu j} + \underbrace{\partial_j g_{0\mu}}_{=0 \text{ for } \mu\neq 0} - \underbrace{\partial_\mu g_{0i}}_{=0})$$

$$= \frac{1}{2} a(t)^{-2} \gamma^{i\mu} \partial_0 (a(t)^2 \gamma_{\mu j})$$

$$= \gamma^{i\mu} \gamma_{\mu j} \frac{1}{2} a(t)^{-2} 2a(t)\dot{a}(t)$$

$$= \delta^i_j \frac{\dot{a}}{a}$$

$$R_{00} = \partial_\lambda \underbrace{\Gamma^\lambda_{00}}_{=0} - \partial_0 \underbrace{\Gamma^\lambda_{0\lambda}}_{=0 \text{ for } \lambda=0} + \underbrace{\Gamma^\lambda_{00}}_{=0} \Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{0\sigma} \Gamma^\sigma_{\lambda 0}$$

$$= -\partial_0 \delta^i_i \frac{\dot{a}}{a} - \delta^i_j \frac{\dot{a}}{a} \delta^j_i \frac{\dot{a}}{a}$$

$$= -3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}$$

Analogous calculations: $R_{0i} = 0$, $\quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2k)\gamma_{ij}$

$$\Rightarrow R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{ij} R_{ij}$$

$$= -3\frac{\ddot{a}}{a} - a^{-2} \underbrace{\gamma^{ij}\gamma_{ij}}_{=\delta^i_i = 3} (\ddot{a}a + 2\dot{a}^2 + 2k)$$

$$= -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

$$\Rightarrow G_{00} = R_{00} - \frac{1}{2} g_{00} R = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

What about $T_{\mu\nu}$?

Consider universe filled with homogeneous fluid with energy density $\rho(t)$ and pressure $p(t)$ (good approximation on large scales)

Consider "cosmic rest frame": Centre-of-mass of fluid at rest

$$\Rightarrow u^\mu = (1, 0, 0, 0)$$

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu - g_{\mu\nu} p = \begin{pmatrix} \rho & & & \\ & a^2 p & & \\ & & a^2 p & \\ & & & a^2 p \end{pmatrix}$$

$\Rightarrow$ For $\mu = \nu = 0$ Einstein eq. becomes

See note on page 8

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \rho - \frac{k}{a^2}} \quad \text{(Friedmann eq.)}$$

$\hookrightarrow$ Relates 2 unknown functions of t: $a(t)$, $\rho(t)$

Need additional eq. to determine $\rho(t)$

E-p conservation: $\nabla_\mu T^{\mu\nu} = 0$

$$\Rightarrow \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma} = 0$$

Consider $\nu = 0$

$$0 = \partial_0 T^{00} + \Gamma^\mu_{\mu 0} T^{00} + \underbrace{\Gamma^0_{00}}_{=0} T^{00} + \underbrace{\Gamma^0_{0j}}_{=0} T^{0j} + \underbrace{\Gamma^0_{i0}}_{=0} T^{i0} + \Gamma^0_{ij} T^{ij}$$

$g^{ik} g^{jl} T_{kl}$

$$= \dot{\rho} + 3\frac{\dot{a}}{a}\rho + \dot{a}a \underbrace{\gamma_{ij} g^{ik}}_{=-a^{-2}\delta_j^k} g^{jl} (-g_{kl} p)$$

$$\Rightarrow \boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0} \quad \text{(E-p conservation)}$$

Note: $\rho(t)$ and $p(t)$ related by eq. of state (eos) of the fluid:

$$p = p(\rho)$$

$\Rightarrow$ For given eos, evolution of universe fully determined by Friedmann eq. + E-p cons.

ij - component of Einstein eq. gives

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G p - \frac{k}{a^2} \quad \text{(automatically satisfied)}$$

4.1 Simple cosmological solutions ($k = 0$)

Example 1: Non-rel. matter (Einstein-de Sitter universe)

EoS: $p \sim m v^2 \approx 0$

$$\Rightarrow \quad \frac{\dot{\rho}}{\rho} = 3\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho = \frac{\text{const}}{a^3}$$

Interpretation: Conservation of particle number N

$$\hookrightarrow \quad n = \frac{N}{V} \sim a^{-3} \quad \Rightarrow \quad \rho = m \cdot n \sim a^{-3}$$

Friedmann eq.: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{\text{const}}{a^3} \quad \Rightarrow \quad \sqrt{a}\, da = \text{const}\, dt$

$$\Rightarrow \quad a^{3/2} = \text{const}\,(t - t_s)$$

$$\Rightarrow \quad a(t) = \text{const}\,(t - t_s)^{2/3}$$

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{2}{3(t - t_s)}$$

For $t \to t_s$: $a \to 0$, $H \to \infty$

$\hookrightarrow$ singularity ("Big Bang")

$\hookrightarrow$ convention: $t_s = 0$

For $t \to \infty$: $a \to \infty$, $H \to 0$

$\hookrightarrow$ Universe keeps expanding forever, but expansion slows down

The age of a matter-dominated universe is

$$t_0 = \frac{2}{3H_0} \sim 10^{10}\ \text{yr}$$

$\uparrow$ inferred from Hubble's law

Present-day density: $\rho_0 = \frac{3}{8\pi G} H_0^2 \approx 10^{-29} \frac{g}{cm^3}$

<u>Example 2:</u> Rel. matter (radiation)

$$T^\mu{}_\mu = 0 \iff \rho - 3p = 0$$

$$\Rightarrow \quad \text{EoS:} \quad p = \frac{1}{3}\rho$$

$$\Rightarrow \quad \frac{\dot{\rho}}{\rho} = 4\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho = \frac{\text{const}}{a^4}$$

Interpretation: Energy of each particle redshifts $\sim \frac{1}{a}$

$$\hookrightarrow \rho = E \cdot n \sim \frac{1}{a} \cdot \frac{1}{a^3} = \frac{1}{a^4}$$

Friedmann eq.: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{\text{const}}{a^4} \Rightarrow a(t) = \text{const} \cdot t^{1/2}$

$$\Rightarrow H(t) = \frac{1}{2t}$$

Note: If the radiation has a thermal (black-body) spectrum, we can define its temperature

$$\rho = \frac{\pi^2}{30} g T^4 \qquad (g: \text{degrees of freedom})$$

$$\Rightarrow \quad T \sim \frac{1}{a} \sim (1+z)$$

Useful relation: $$\boxed{H = \sqrt{\frac{8\pi^3}{90}}\sqrt{g}\,\frac{T^2}{M_{pl}} \approx 1.66\sqrt{g}\,\frac{T^2}{M_{pl}}}$$

with $M_{pl} = G^{-1/2}$

Example 3: Vacuum energy

Assume that vacuum has non-vanishing energy density $T_{\mu\nu} = \rho\, g_{\mu\nu}$

$\Rightarrow \quad p = -\rho \qquad$ (negative pressure)

$\Rightarrow \quad \dot{\rho} = 0$

Interpretation: Vacuum energy does not dilute as space expands

Friedmann eq.: $\quad \frac{\dot{a}}{a} = \text{const} \quad \Rightarrow \quad a = \text{const}\; e^{Ht}$

with $H = \sqrt{\frac{8\pi}{3} G \rho}$

Very different from examples 1 + 2:

- All quantities finite for $t \to -\infty$ $\Rightarrow$ No singularity
- $\ddot{a} > 0 \quad \Rightarrow$ Accelerated expansion

Example 4: General component with $p = w\varrho$ $(w > -1)$

$$\Rightarrow \quad a = \text{const} \cdot t^{2/3(1+w)}$$

↳ Decelerated expansion for $w > -\frac{1}{3}$
Accelerated " " $w < -1/3$

Comment: Possible to generalise Einstein eq. to

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

cosmological constant

↳ First introduced by Einstein to guarantee stationary universe $(H = 0)$

↳ Hubble's law: $H \neq 0$
Einstein: "Größte Eselei meines Lebens"

Modern interpretation: Λ contributes to vacuum energy

$$\varrho_{vac} \rightarrow \varrho_{vac} + \frac{\Lambda}{8\pi G}$$

But: So far no successful prediction of ϱ_{vac} from first principles

→ "Cosmological constant problem"

Conventions: $\rho_{vac} \equiv \rho_\Lambda$

vacuum energy $\equiv$ dark energy

$k = 1 \iff$ closed universe

$k = 0 \iff$ flat universe

$k = -1 \iff$ open universe

Note: $T_\mu{}^\nu = (p + \rho) u_\mu u^\nu - \delta_\mu{}^\nu p$

$$= \begin{pmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix}$$

$\Rightarrow$ same as in Minkowski space

But $T_{\mu\nu} = g_{\nu\rho} T_\mu{}^\rho = \begin{pmatrix} \rho & & & \\ & a^2 p & & \\ & & a^2 p & \\ & & & a^2 p \end{pmatrix}$

$\Rightarrow$ different from Minkowski space