4.2 The ΛCDM model

In general, the energy density of the universe is a sum of different components

$$\rho_{tot} = \rho_M + \rho_{rad} + \rho_\Lambda$$

$$\Rightarrow \quad H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \left(\rho_M + \rho_{rad} + \rho_\Lambda\right) - \frac{k}{a^2}$$

Define $\rho_c = \frac{3}{8\pi G} H_0^2$ (critical density)

$$\Omega_i = \frac{\rho_{i,0}}{\rho_c} \quad \text{(present-day abundance)}$$

$$\Omega_{curv} = -\frac{k}{a^2 H_0^2}$$

$$\Rightarrow \quad \Omega_M + \Omega_{rad} + \Omega_\Lambda + \Omega_{curv} = 1$$

Note: $\Omega_M + \Omega_{rad} + \Omega_\Lambda = 1 \iff \rho_{tot} = \rho_c$

$\iff k = 0$ (flat universe)

Observations yield $\Omega_M + \Omega_\Lambda \approx 1$

$\Rightarrow \Omega_{rad}, \Omega_{curv} \ll 1$

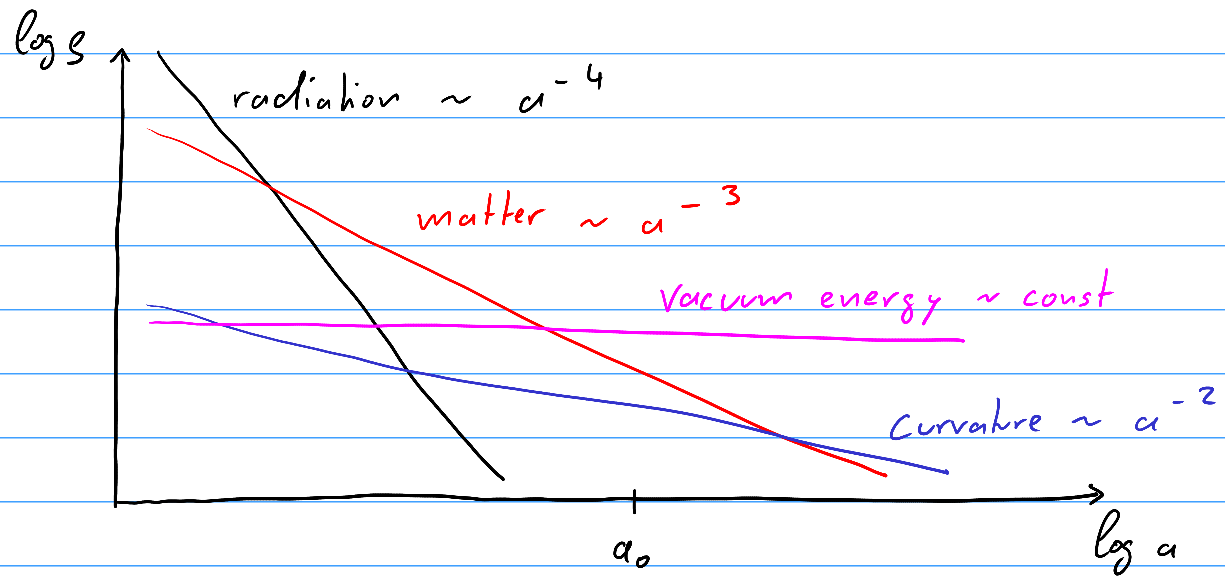
Lower bound on Ω_{rad} from CMB:

$$\rho_{rad,0} \geq \rho_{\gamma,0} = 2\frac{\pi^2}{30} \underbrace{T_0}_{\approx 2.726\,K}{}^4 = 2.6 \cdot 10^{-10} \frac{GeV}{cm^3}$$

$$\rho_c \approx 5 \cdot 10^{-6} \frac{\text{GeV}}{\text{cm}^3} \Rightarrow \Omega_{rad} \gtrsim 5 \cdot 10^{-5}$$

Using $\rho_m \sim a^{-3}$, $\rho_{rad} \sim a^{-4}$, $\rho_\Lambda \sim \text{const}$:

$$\boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G \rho_c \left[\underbrace{\Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_{rad}\left(\frac{a_0}{a}\right)^4}_{\text{dominate for small } a} + \underbrace{\Omega_\Lambda + \Omega_{curv}\left(\frac{a_0}{a}\right)^2}_{\text{dominate for large } a} \right]}$$



For $a \approx a_0$, we can neglect Ω_{rad}, Ω_{curv}

$$\Rightarrow \dot a^2 = \frac{8\pi}{3} G \rho_c \left(\frac{\Omega_m a_0^3}{a} + \Omega_\Lambda a^2 \right) \qquad (*)$$

$$\ddot a = a \frac{4\pi}{3} G \rho_c \left(2\Omega_\Lambda - \Omega_m \left(\frac{a_0}{a}\right)^3 \right)$$

Transition from decelerated ($\ddot a < 0$) to accelerated ($\ddot a > 0$) expansion at

$$\left(\frac{a_0}{a_{ac}}\right)^3 = \frac{2\Omega_\Lambda}{\Omega_m}$$

Realistic values ($\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$)

$$z_{ac} = \left(\frac{2\Omega_\Lambda}{\Omega_m}\right)^{1/3} - 1 \approx 0.76 \qquad \text{(pretty recent!)}$$

For $a \ll a_0$, we can neglect Ω_Λ, Ω_{curv}

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \left(\frac{a_0}{a}\right)^3 \left[\Omega_m + \Omega_{rad}\frac{a_0}{a}\right]$$

$\Rightarrow$ Matter-radiation equality: $\dfrac{a_0}{a_{eq}} = \dfrac{\Omega_m}{\Omega_{rad}} \sim 10^4$

More accurate estimate: $\rho_{rad} = \rho_\gamma + \rho_\nu \approx 1.7\,\rho_\gamma$

will derive this later!

$$\Rightarrow \quad 1 + z_{eq} = \frac{a_0}{a_{eq}} \approx 3 \cdot 10^3$$

$$T_{eq} = (1 + z_{eq})\,T_0 \approx 0.75\,\text{eV}$$

For $a < a_{eq}$, universe is radiation dominated

$$\Rightarrow \quad t_{eq} \approx \frac{1}{2H_{eq}} \approx 7 \cdot 10^4\ \text{yr}$$

For $a \gg a_{eq}$:

$$(*) \Rightarrow \quad a(t) = a_0 \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\sqrt{\Omega_\Lambda}\,H_0 t\right)$$

$$\sim \begin{cases} t^{2/3} & \text{for } \frac{3}{2}\sqrt{\Omega_\Lambda}\,H_0 t \ll 1 \\ e^{\sqrt{\Omega_\Lambda}\,H_0 t} & \text{for } \frac{3}{2}\sqrt{\Omega_\Lambda}\,H_0 t \gg 1 \end{cases}$$

$$\left(\frac{a_0}{a(t_{ac})}\right)^3 = \frac{2\Omega_\Lambda}{\Omega_M} \quad \Rightarrow \quad t_{ac} \sim 7.3 \text{ Gyr}$$

$$\frac{a_0}{a(t_0)} = 1 \quad \Rightarrow \quad t_0 \sim 13.5 \text{ Gyr}$$

Because of recent accelerated expansion, age of universe is larger than for $\Omega_M = 1$

↳ Consistent with observation of ancient stars requiring $t_0 > 13$ Gyr

⇒ Evidence for $\Omega_\Lambda > 0$

How to obtain more accurate estimates of Ω_i?

4.3 Brightness-redshift relation

To measure expansion history, need far-away objects of known absolute luminosity ("standard candles")

Example: Type Ia supernovae (SNe Ia)

↳ Thermonuclear explosion of a white dwarf in a binary system

↳ Known relation between peak luminosity and time-dependence of emission

Need to relate absolute luminosity $L = \frac{\text{emitted energy}}{\text{time}}$ to observed brightness J

$$J = \frac{\#\text{ photons} \cdot \text{observed energy}}{\text{time} \cdot \text{area}}$$

Consider photon emitted at t_i and observed at t_o.

$$\text{observed energy} = \text{emitted energy} \cdot \frac{a(t_i)}{a_0}$$

$$\frac{\#\text{ photons}}{\text{time}} = \frac{\#\text{ emitted photons}}{\text{time}} \cdot \frac{a(t_i)}{a_0}$$ ← redshift

To calculate the area, use

$$ds^2 = dt^2 - a(t)^2 \left[d\chi^2 + S_K^2(\chi) d\Omega^2 \right] = 0$$

$$= \begin{cases} \sin \chi & K = 1 \\ \chi & K = 0 \\ \sinh \chi & K = -1 \end{cases}$$

$$\Rightarrow \chi(t_i) = \int_{t_i}^{t_0} \frac{dt}{a(t)}$$

$$z(t) = \frac{a_0}{a(t)} - 1 \Rightarrow dz = -\frac{a_0}{a(t)^2} \dot{a}(t)\, dt = -\frac{a_0}{a(t)} H(z) dt$$

$$\chi(z) = \int_0^z \frac{dz'}{a_0 H(z')} \approx \int_0^z \frac{dz'}{a_0 H_0} \frac{1}{\sqrt{\Omega_m (z'+1)^3 + \Omega_\Lambda + \Omega_{curv} (z'+1)^2}}$$

At $t = t_0$ the photons pass through a sphere of size

$$S(z) = 4\pi d^2(z) = 4\pi a_0^2 S_K^2(\chi(z))$$

$$\Rightarrow \quad \boxed{J = \frac{L}{(1+z)^2\, S(z)} = \frac{L}{4\pi r_L^{\,2}}}$$

with $r_L = (1+z)\, a_0\, S_k(\chi(z))$ ("luminosity distance")

Comments:

- For $z \ll 1$: $(z'+1) \approx 1 \Rightarrow \chi(z) \approx \dfrac{z}{a_0 H_0}$

$$\Rightarrow d(z) \approx \frac{z}{H_0} \quad \text{(Hubble's law)}$$

- Consider $\Omega_{curv} = 0 \Rightarrow \Omega_m + \Omega_\Lambda = 1$

$$\Rightarrow H_0\, d(z) = \int_0^z \frac{dz'}{\sqrt{\Omega_m (1+z')^3 + (1-\Omega_m)}}$$

$$= \int_0^z \frac{dz'}{\sqrt{\Omega_m (3z' + 3z'^2 + z'^3) + 1}}$$

$$= \begin{cases} 2\left(1 - \dfrac{1}{\sqrt{1+z}}\right) & \Omega_m = 1,\ \Omega_\Lambda = 0 \\ z & \Omega_\Lambda = 1,\ \Omega_m = 0 \end{cases}$$

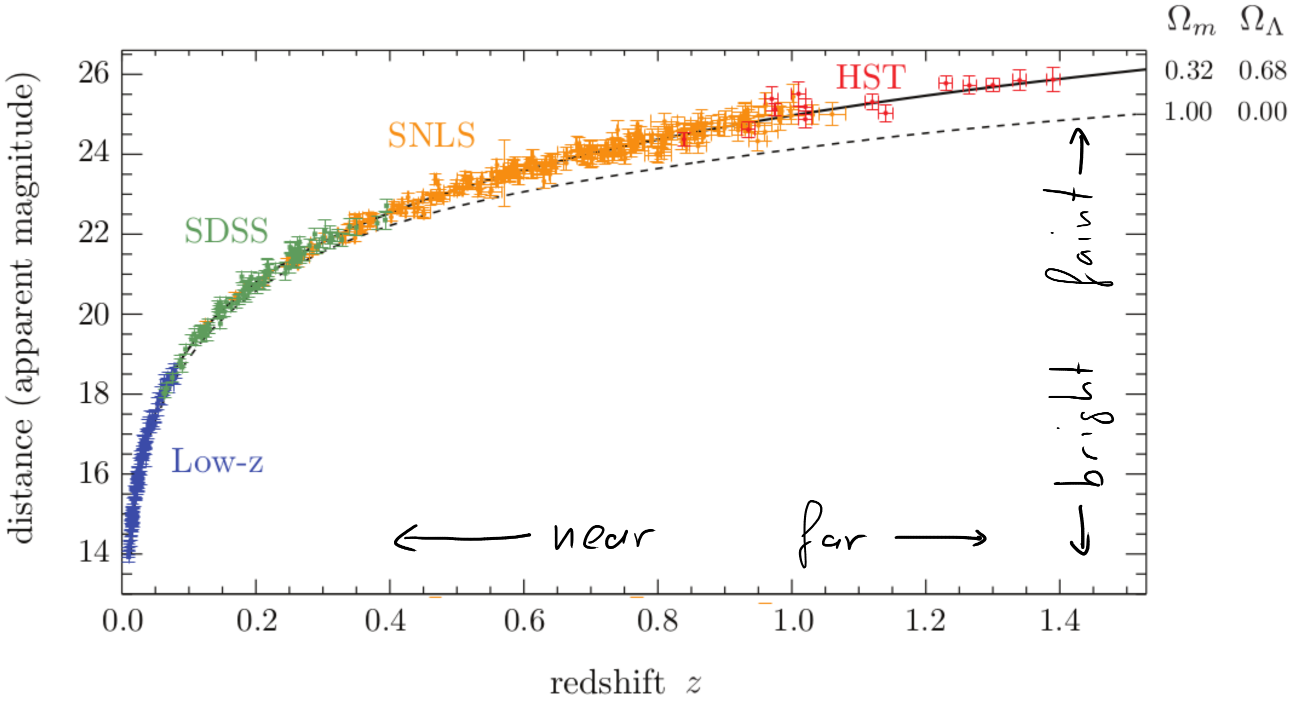
$\Rightarrow d(z)$ increases with decreasing Ω_m

$\Rightarrow J$ decreases " " "

$\hookrightarrow$ vacuum energy makes standard candles less bright

$m = 5\log r_{ph}(z) + \text{const}$

Exactly what is observed!



→ Nobel prize 2011

- For $\Omega_{curv} > 0$ $(k=-1)$ we obtain

$$\chi(z) \overset{z \ll 1}{\approx} \int_0^z \frac{dz'}{a_0 H_0} \frac{1}{\sqrt{(1+3z')\Omega_m + \Omega_\Lambda + (1+2z')\Omega_{curv}}}$$

$$\approx \int_0^z \frac{dz'}{a_0 H_0} \left(1 - \tfrac{1}{2}(3\Omega_m + 2\Omega_{curv})\, z'\right)$$

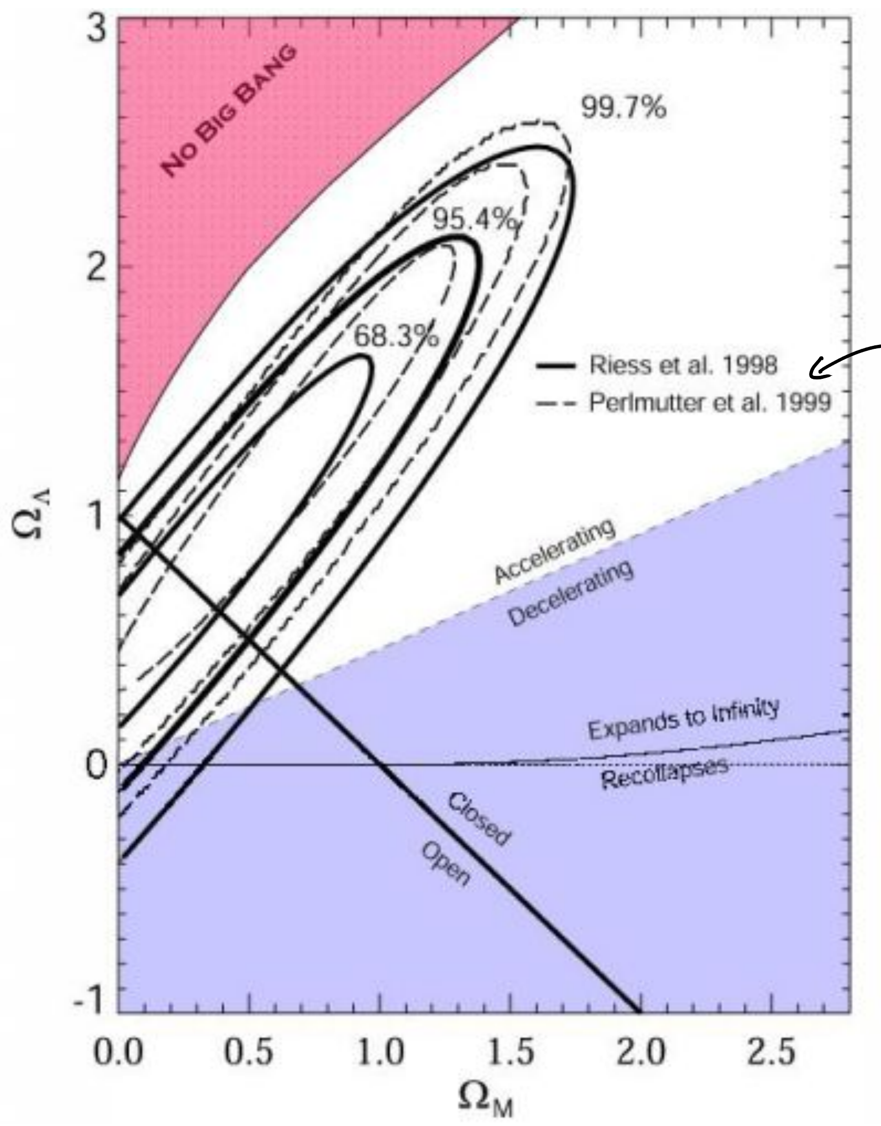
$$= \frac{1}{a_0 H_0}\left(z - \frac{z^2}{4}(3\Omega_m + 2\Omega_{curv}) + \mathcal{O}(z^3)\right)$$

$$= \frac{1}{a_0 H_0}\left(z - \frac{z^2}{4}(2 + \Omega_m - 2\Omega_\Lambda) + \mathcal{O}(z^3)\right)$$

↳ Non-linear correction to Hubble's law

Data clearly requires $\Omega_M - 2\Omega_\Lambda > 0$

=> Present universe experiences accelerated expansion!



very old data!
modern constraints
much tighter!