§5 Early Universe Thermodynamics

So far: Treated matter and radiation as non-interacting perfect fluids

More realistic: Ensembles of interacting particles

↳ Sufficiently strong interactions ⇒ local thermal equilibrium (LTE)
will quantify this next lecture!

Each particle species i characterised by distribution function

$$f_i(\vec{p}) = \frac{1}{(2\pi)^3} \frac{1}{e^{(E_i - \mu_i)/T} \mp 1} \qquad \begin{array}{l} - : \text{boson} \\ + : \text{fermion} \end{array}$$

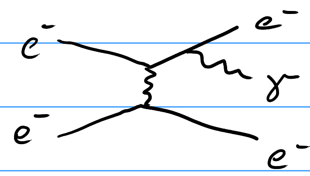
with $E_i = \sqrt{\vec{p}^{\,2} + m_i^2}$

T : temperature (common for all species)
μ_i : chemical potential (may depend on T)

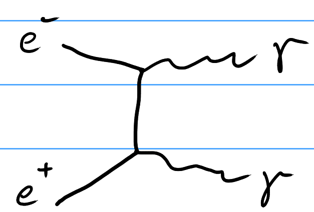
For process $A_1 + A_2 + \ldots \longleftrightarrow B_1 + B_2 + \ldots$ in chemical equil.:

$$\mu_{A_1} + \mu_{A_2} + \ldots = \mu_{B_1} + \mu_{B_2} + \ldots$$

Examples : • $e^- + e^- \rightarrow e^- + e^- + \gamma$



$\Rightarrow \mu_\gamma = 0$



• $e^- + e^+ \rightarrow 2\gamma$

$\Rightarrow \mu_{e^+} = -\mu_{e^-}$

For $m_i \gg T, \mu_i$: $E_i \approx m_i + \frac{1}{2}\frac{\vec{p}^2}{m_i}$

$$\Rightarrow f_i(\vec{p}) \approx \frac{1}{(2\pi)^3} e^{(\mu_i - m_i)/T} e^{-\vec{p}^2/2m_iT}$$

For given $f_i(\vec{p})$ can calculate

- $n_i = g_i \int f_i(\vec{p})\, d^3p \overset{EdE=pdp}{=} 4\pi g_i \int f_i(E) \sqrt{E^2 - m_i^2}\, E\, dE$

(number density)

- $\rho_i = g_i \int f_i(\vec{p})\, E_i(\vec{p})\, d^3p = 4\pi g_i \int f_i(E) \sqrt{E^2 - m_i^2}\, E^2\, dE$

(energy density)

- $p_i = \frac{g_i}{3} \int f_i(\vec{p}) \frac{\vec{p}^2}{E_i(\vec{p})}\, d^3p = \frac{4\pi g_i}{3} \int f_i(E) (E^2 - m_i^2)^{3/2}\, dE$

(pressure)

g_i : degrees of freedom

SM: $i =$	γ	e^-	e^+	Z	W^-	W^+	ν	$\bar{\nu}$	h	q	$\bar{q}$	g
$g_i =$	2	2	2	3	3	3	1	1	1	6	6	16

$$\sum_i g_i = \sum_{\text{bosons}} g_i + \sum_{\text{fermions}} g_i = (2 + 3\times 3 + 1 + 16) + (3 \times 2 \times (2 + 1 + 6 + 6))$$

$$= 28 + 90 = 118$$

5.1 Relativistic species

Assume $T \gg m_i$, $\mu_i = 0$

$$\Rightarrow \rho_i = \frac{g_i}{2\pi^2} \int \frac{E^3}{e^{E/T} \mp 1} dE = \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{boson} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{fermion} \end{cases}$$

For several relativistic species

$$\boxed{\begin{aligned} \rho &= \sum_i \rho_i = g_* \frac{\pi^2}{30} T^4 \\ \text{with } g_* &= \sum_{\text{rel. bosons}} g_i + \frac{7}{8} \sum_{\text{rel. fermions}} g_i \end{aligned}}$$

(effective number of rel. degrees of freedom)

Examples: • $T \gg m_t$: $g_* = 28 + \frac{7}{8} \cdot 90 = 106.75$

• $m_\mu \gg T \gg m_e$: $g_* = 2 + \frac{7}{8}(2 \times 2 + 3 \times 2 \times 1)$

$$= 10.75$$

$$p_i = \frac{g_i}{6\pi^2} \int \frac{E^3}{e^{E/T} \mp 1} dE = \frac{\rho_i}{3} \quad \text{(as expected)}$$

$$n_i = \frac{g_i}{2\pi^2} \int \frac{E^2}{e^{E/T} \mp 1} dE = \begin{cases} g_i \frac{\zeta(3)}{\pi^2} T^3 & \text{boson} \\ \frac{3}{4} g_i \frac{\zeta(3)}{\pi^2} T^3 & \text{fermion} \end{cases}$$

$$\Rightarrow \langle E \rangle = \frac{\rho_i}{n_i} = \begin{cases} 2.70\, T & \text{boson} \\ 3.15\, T & \text{fermion} \end{cases}$$

$\zeta(3) \approx 1.2$

5.2 Non-relativistic species

$$n_i = \frac{g_i}{2\pi^2} e^{\frac{\mu_i - m_i}{T}} \int \underbrace{e^{-p^2/2m_iT} p^2}_{\text{Maxwell-Boltzmann distribution}} dp$$

$$= g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\frac{\mu_i - m_i}{T}}$$

For $\mu_i = 0$ density of non-rel. particles is exponentially suppressed (Boltzmann suppression)

Interpretation: Annihilation process $h + h \to \ell + \ell$ always possible (h: heavy, ℓ: light)

Production process $\ell + \ell \to h + h$ requires $E > 2m_h$

↳ Exponentially unlikely for $T \ll m_h$

↳ Heavy particles "annihilate away"

↳ Energy density dominated by light species (unless $\mu_i \neq 0$)

$$\rho_i = m_i \cdot n_i + \frac{3}{2} n_i T \overset{T \to 0}{\approx} m_i n_i$$

$$p_i = T n_i \overset{T \to 0}{\approx} 0$$

(Note: $pV = NT$ ideal gas law for $k_B = 1$)

5.3 Entropy

First law of thermodynamics:

$$dE = T\underset{\substack{\uparrow \\ \text{entropy}}}{dS} - p\,dV + \sum_i \mu_i\, dN_i$$

Define $s = \frac{S}{V}$ (entropy density)

$\Rightarrow ds = \frac{dS}{V} - s\frac{dV}{V}$ (analogous for $\rho = \frac{E}{V}$, $n = \frac{N}{V}$)

$$\Rightarrow \left(Ts - p - \rho + \sum_i \mu_i n_i\right) dV + \left(T ds - d\rho + \mu\, dn\right) V = 0$$

Consider $V = \text{const}$ $\Rightarrow dV = 0$

$$\Rightarrow T ds - d\rho + \mu\, dn = 0$$

For arbitrary volume $\Rightarrow Ts - p - \rho + \sum_i \mu_i n_i = 0$

$$\Rightarrow \boxed{s = \frac{p + \rho - \sum_i \mu_i n_i}{T}}$$

Example: • Rel. species with $\mu_i = 0$

$$\Rightarrow s_i = \frac{p_i + \rho_i}{T} = \frac{4}{3}\frac{\rho_i}{T} = \begin{cases} g_i \frac{2\pi^2}{45} T^3 & \text{boson} \\ \frac{7}{8} g_i \frac{2\pi^2}{45} T^3 & \text{fermion} \end{cases}$$

$$\Rightarrow s = \sum_i s_i = g_* \frac{2\pi^2}{45} T^3$$

- Non-rel. species

$$s_i = \frac{\rho_i + p_i - \mu_i n_i}{T}$$

$$= \frac{m_i n_i + \frac{3}{2} n_i T + n_i T - \mu_i n_i}{T}$$

$$= n_i \left(\frac{5}{2} + \underbrace{\frac{m_i - \mu_i}{T}}_{= \log\left[\frac{g_i}{n_i}\left(\frac{m_i T}{2\pi}\right)^{3/2}\right]} \right)$$

=> Similar Boltzmann suppression as for n_i

Second law of thermodynamics : $dS = 0$ for equil. evolution

Proof (assuming $\sum_i \mu_i dN_i = 0$):

$$T dS = p dV + d(\rho \cdot V) = (p+\rho) dV + V d\rho$$

Remember: $V \sim a^3 \Rightarrow dV = 3a^2 da = 3V \frac{da}{a}$

$$\Rightarrow T \frac{dS}{dt} = V \Big(\underbrace{3(p+\rho)\frac{\dot{a}}{a} + \dot{\rho}}_{=0 \ (E\text{-}p \text{ conservation})} \Big)$$

$\Rightarrow s \cdot a^3 = \text{const}$

=> entropy density convenient measure of expansion

Define $Y_i = \frac{n_i}{s} \sim n_i \cdot V = N_i$

If no particles are produced/destroyed $\Rightarrow Y_i = \text{const}$

Example: Baryon number conservation: $N_B - N_{\bar{B}} = \text{const}$

$$\Rightarrow \Delta_B = \frac{n_B}{s} - \frac{n_{\bar{B}}}{s} = \text{const}$$

Particle thresholds

Shown before that $T \sim a^{-1}$ during RD

Implicitly assumed $g_* = \text{const}$

More accurate: $g_* T^3 a^3 = \text{const} \Rightarrow T \sim g_*^{-1/3} a^{-1}$

If T drops below m_i

$\Rightarrow$ species becomes non-rel.

$\Rightarrow$ g_* decreases

$\Rightarrow$ T decreases more slowly

Interpretation: As non-rel. particles annihilate away, entropy transferred to rel. species