§6 Boltzmann equation

Last time: Assumed all species to be in equil.

↳ Not always satisfied

↳ Departure from equil. essential for cosmology

Today: General evolution of $f(p,t)$ ← homogeneos & isotropic

↳ Can be written as

$$L[f] = C[f]$$

Liouville operator → phase space evolution

Collision operator → effect of interactions

For $C[f]=0$: Particle number conserved

=> Phase space volume conserved

$$\Rightarrow \frac{df(p,t)}{dt} = 0 = \frac{\partial f}{\partial t} + \frac{dp}{dt}\frac{\partial f}{\partial p}$$

Consider particle 4-mom. $p^\mu = (E, \vec{p})$

$$p\,dp = E\,dE = p^0 dp^0$$

$$\Rightarrow p\frac{dp}{dt} = p^0\frac{dp^0}{dt} = -\Gamma^0_{\alpha\beta}p^\alpha p^\beta = -H(t)\,p^2$$

geodesic eq.

$$\Rightarrow \quad L[f] = \frac{\partial f}{\partial t} - H(t)\, p \frac{\partial f}{\partial p} = \frac{\partial f}{\partial t} - H(t) \frac{p^2}{E} \frac{\partial f}{\partial E}$$

Often convenient to consider integral

$$\frac{g}{(2\pi)^3} \int d^3p \; L[f] = \frac{\partial}{\partial t} \underbrace{\left(\frac{g}{(2\pi)^3} \int d^3p \; f \right)}_{= n(t)} - H \frac{g}{(2\pi)^3} \underbrace{\int d^3p \; p \frac{\partial f}{\partial p}}_{= -4\pi \int_0^\infty 3p^2 f \, dp}$$

$$= \dot{n} + 3Hn = \frac{1}{a^3} \frac{d}{dt} (n a^3)$$

↳ Liouville operator describes change in $n(t)$ due to expansion

To find explicit form for $C[f]$ consider interaction $1 + 2 \leftrightarrow 3 + 4$

↳ Increase in f_1 proportional to

$$\sum_{\text{spins}} |\mathcal{M}_{34 \to 21}|^2 f_3 f_4 \underbrace{(1 \pm f_1)(1 \pm f_2)}$$

reaction probability

density of initial states

boson: + (Bose enhancement)
fermion: − (Pauli blocking)

↳ Decrease in f_1 proportional to

$$\sum_{\text{spins}} |\mathcal{M}_{12 \to 34}|^2 f_1 f_2 (1 \pm f_3)(1 \pm f_4)$$

Simplifications:

- Often possible to assume $f \ll 1 \Rightarrow (1 \pm f) \approx 1$
- For most processes $|\mathcal{M}_{12\to34}|^2 = |\mathcal{M}_{34\to12}|^2 \equiv |\mathcal{M}|^2$
- Need to integrate over all possible momenta

$$\Rightarrow C[f_1] = \frac{1}{2E_1} \int d\Pi_2 \, d\Pi_3 \, d\Pi_4 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$d\Pi = \frac{d^3p}{(2\pi)^3 2E} \qquad \times \sum_{\text{spins}} |\mathcal{M}|^2 (f_3 f_4 - f_1 f_2)$$

Additional assumption: f_3 and f_4 given by equil. dist.

$$\Rightarrow f_3 \cdot f_4 = f_3^{eq} \cdot f_4^{eq} = e^{-(E_3+E_4)/T}$$

$$\overset{E\text{ cons.}}{=} e^{-(E_1+E_2)/T} = f_1^{eq} f_2^{eq}$$

$$\Rightarrow C[f_1] = \frac{1}{2E_1} \int d\Pi_2 \, (f_1^{eq} f_2^{eq} - f_1 f_2)$$

$$\times \underbrace{\int d\Pi_3 \, d\Pi_4 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \sum_{\text{spins}} |\mathcal{M}|^2}_{= \sigma v}$$

$$\Rightarrow \frac{g}{(2\pi)^3} \int d^3p \, C[f_1] = \int d\Pi_1 \, d\Pi_2 \, (f_1^{eq} f_2^{eq} - f_1 f_2) \, \sigma v$$

Now assume $\frac{n}{n^{eq}} = \frac{f}{f^{eq}}$

$$\Rightarrow \frac{g}{(2\pi)^3}\int d^3p\, C[f_1] = (n_1^{eq}n_2^{eq} - n_1 n_2)\cdot \underbrace{\frac{1}{n_1^{eq}n_2^{eq}}\int d\Pi_1\, d\Pi_2\, f_1^{eq} f_2^{eq}\, \sigma v}_{\equiv \langle \sigma v\rangle}$$

"thermally averaged cross section"

$$\Rightarrow \boxed{\dot{n}_1 + 3Hn_1 = \langle\sigma v\rangle (n_1^{eq}n_2^{eq} - n_1 n_2)}$$

(Boltzmann eq.)

Example: Consider $e^+e^- \leftrightarrow \gamma\gamma$ with $n_{e^+} = n_{e^-} \equiv n$

$$\Rightarrow \dot{n} + 3Hn = \langle\sigma v\rangle \left[(n^{eq})^2 - n^2\right]$$

Using $Y = \frac{n}{s}$ and $3Hs + \dot{s} = 0$

$$\Rightarrow \dot{n} = \dot{Y}s + Y\dot{s} = \dot{Y}s - 3HsY$$

$$\Rightarrow \dot{Y} = -\langle\sigma v\rangle s\,(Y^2 - Y_{eq}^2)$$

Define $x \equiv \frac{m}{T}$ $\Rightarrow \frac{ds}{dx} = \dot{s}\frac{dt}{dx}$

$$= -3Hs\frac{dt}{dx}$$

$$\Rightarrow \frac{dY}{dx} = \frac{1}{3H}\frac{ds}{dx}\langle\sigma v\rangle (Y^2 - Y_{eq}^2)$$

For $g_* = \text{const}$: $s \sim T^3$

$$\Rightarrow \quad \frac{ds}{dx} = 3\frac{s}{T}\frac{dT}{dx} = -\frac{3s}{x}$$

$$\Rightarrow \quad \boxed{\frac{dY}{dx} = -\frac{s}{Hx}\langle\sigma v\rangle\left(Y^2 - Y_{eq}^2\right)}$$

Interpretation: $\sigma \cdot \underbrace{v \cdot n}_{\text{particle flux}} = \Gamma$ "interaction rate"

↳ determines how rapid an interaction happens

$$\Rightarrow \quad \underbrace{\frac{x}{Y_{eq}}\frac{dY}{dx}}_{\substack{\text{rel. change} \\ \text{in density}}} = -\frac{\Gamma}{H}\underbrace{\left(\frac{Y^2}{Y_{eq}^2} - 1\right)}_{\text{departure from equil.}}$$

interaction

- If $Y < Y_{eq}$: $\frac{dY}{dx} > 0 \Rightarrow$ increase
- If $Y > Y_{eq}$: $\frac{dY}{dx} < 0 \Rightarrow$ decrease

evolution towards equil.

- For $\Gamma \gg H$: Quick evolution $Y \to Y_{eq}$
  $\Rightarrow$ therm. equil.
- For $\Gamma \ll H$: $\frac{dY}{dx} \to 0 \Rightarrow$ no therm. equil.

  $\Rightarrow$ comov. number density constant

For $e^+e^- \leftrightarrow \gamma\gamma$ and $E \gg m_e$: $\sigma \sim \frac{\alpha^2}{E^2}$

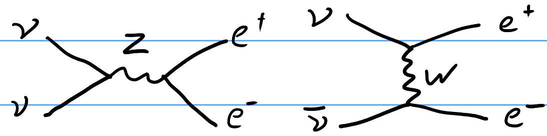
$\langle \sigma v \rangle \sim \frac{\alpha^2}{T^2}$ and $n_{eq} \sim T^3$ $\Rightarrow$ $\Gamma \sim \alpha^2 T$

$\Rightarrow \frac{\Gamma}{H} \sim \frac{\alpha^2 M_P}{T} \sim \frac{10^{15}\,\text{GeV}}{T} \gg 1$ for all relevant T

$\Rightarrow$ e^+, e^-, γ in therm. equil.

↳ Also true for other particles with EM charge

Next: Consider $\nu\bar{\nu} \leftrightarrow e^+e^-$



For $m_e \ll E \ll m_{W,Z}$: $\sigma \sim G_F^2 E^2$

$G_F = 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$

$\Rightarrow \langle \sigma v \rangle \sim G_F^2 T^2$

$\Rightarrow \Gamma \sim G_F^2 T^5$

$\Rightarrow \frac{\Gamma}{H} \sim G_F^2 T^3 M_P \sim 10^9\,\text{GeV}^{-3}\,T^3 \sim \left(\frac{T}{1\,\text{MeV}}\right)^3$

$\Rightarrow \frac{\Gamma}{H} > 1 \Leftrightarrow T > 1\,\text{MeV}$

$\Rightarrow$ Neutrinos decouple from therm. equil. when T drops below $1\,\text{MeV}$

More detailed calculation: $T_\nu^{dec} = 2\text{-}3\,\text{MeV}$ (dep. on flavour)