§10 Cosmological Perturbation Theory

So far: Universe completely homogeneous

$$ds^2 = dt^2 - a(t)^2 [dx^2 + dy^2 + dz^2]$$

↳ clearly inconsistent with observations

Now: Consider small perturbations

$$g_{\mu\nu}(t) \to \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \vec{x})$$

$$T_{\mu\nu}(t) \to \underbrace{\bar{T}_{\mu\nu}(t)}_{\text{"background"}} + \underbrace{\delta T_{\mu\nu}(t, \vec{x})}_{\text{"perturbations"}}$$

Assumption: Small perturbations $(\delta X \ll X)$
=> Study all eqs. at linear order in δX

General findings: - Perturbations are extremely small initially, but grow with time

=> linear perturbation theory breaks down in late Universe $(\delta X \sim X)$
=> non-linear structure formation (stars, galaxies, ...)

- Hierarchical structure formation

=> small structures form first and become non-linear earlier

For simplicity, we will neglect curvature and write the background metric as

$$ds^2 = a^2(\eta)\left[d\eta^2 - \delta_{ij}\, dx^i dx^j\right] = a^2(\eta)\,\eta_{\mu\nu}\, dx^\mu dx^\nu$$

with $d\eta = \frac{dt}{a(t)}$ (conformal time)

Convention: $\dot{} \;\hat{=}\; \frac{d}{dt}$ $\qquad {}' \;\hat{=}\; \frac{d}{d\eta} = a(\eta)\frac{d}{dt}$

$$H = \frac{\dot{a}}{a} = \frac{a'}{a^2}$$

$$\mathcal{H} = \frac{a'}{a} \quad \text{(conformal Hubble rate)}$$

Background satisfies Friedmann eqs.

$$\mathcal{H}^2 = \frac{8\pi}{3} G_N \bar{\rho}\, a^2$$

$$2\mathcal{H}' + \mathcal{H}^2 = -8\pi G_N \bar{P} a^2$$

and E-p conservation:

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{P})$$

$$\Rightarrow \mathcal{H} \sim \begin{cases} a^{-1} & \text{(RD)} \\ a^{-1/2} & \text{(MD)} \end{cases}$$

$$\Rightarrow a \sim \begin{cases} \eta & \text{(RD)} \\ \eta^2 & \text{(MD)} \end{cases}$$

Useful definitions:

- <u>Particle horizon</u>

Photons travel with $ds^2 = 0 \iff |d\vec{x}| = d\eta$

$\Rightarrow$ Photons emitted at $t = 0$ travel the physical distance

$$\ell_H(t) = a(t)\,\eta(t) = a(t)\int_0^t \frac{dt'}{a(t')}$$

$$= \begin{cases} 1/H(t) & \text{RD} \\ 2/H(t) & \text{MD} \end{cases}$$

$\Rightarrow$ Size of the causally connected region

- <u>Fourier transform</u>

$$f(\eta, \vec{x}) = \int d^3k \; e^{i\vec{k}\cdot\vec{x}} f(\eta, \vec{k})$$

Note: Same symbol!

$\vec{k}$ is called conformal momentum

↳ physical momentum $\vec{q} = \dfrac{\vec{k}}{a(\eta)}$ (redshift)

Will show that perturbations behave fundamentally differently for $|\vec{q}|\ell_H > 1$ and $|\vec{q}|\ell_H < 1$.

10.1 Metric perturbations

Most general perturbed spacetime:

$$ds^2 = a^2(\eta)\Big[(1+2\underbrace{A}_{\text{scalar}})d\eta^2 - 2\underbrace{B_i}_{\text{vector}}dx^i d\eta - (\delta_{ij} + \underbrace{h_{ij}}_{\text{symmetric tensor}})dx^i dx^j\Big]$$

A, B_i, h_{ij} contain $\underbrace{10}_{1+3+6}$ indep. functions of η and $\vec{x}$.

Perform scalar - vector - tensor decomposition:

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{\hat{B}_i}_{\text{divergenceless vector}}$$

$$\partial_i \hat{B}^i = 0$$

$$h_{ij} = 2C\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E + \left(\partial_i\hat{E}_j + \partial_j\hat{E}_i\right) + 2\hat{E}_{ij}$$

scalars $\quad$ divergenceless vector

with $\partial^i \hat{E}_{ij} = 0$ and $\hat{E}_{ij}\delta^{ij} = 0$ (traceless & transverse tensor)

Again, $A, B, C, E, \hat{B}_i, \hat{E}_i, \hat{E}_{ij}$ contain $\underbrace{10}_{4\times 1 + 2\times 2 + 1\times 2}$ indep. functions

Einstein eqs. do not mix scalar, vector + tensor perturbations $\rightarrow$ three indep. sets of eqs.

Vector perturbations play no role in cosmology and can be neglected

Tensor perturbations correspond to gravitational waves (see GR lecture)

↳ Focus on scalar perturbations A, B, C, E for now

Problem: Perturbations not invariant under coordinate (i.e. gauge transformation)

Consider $x^\mu \to \tilde{x}^\mu \equiv x^\mu + \xi^\mu(\eta, \vec{x})$

$$\xi^0 \equiv T, \quad \xi^i \equiv L^i = \partial^i L + \hat{L}^i$$

Use $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = \tilde{g}_{\alpha\beta}(\tilde{x}) d\tilde{x}^\alpha d\tilde{x}^\beta$

$$\Rightarrow \quad g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(\tilde{x})$$

Assume ξ^μ is small and expand to linear order in perturbations

Example: $\mu = \nu = 0$

$$a^2(\eta)(1+2A) = \left(\frac{\partial \tilde{\eta}}{\partial \eta}\right)^2 a^2(\eta + T)(1 + 2\tilde{A})$$

$$= (1 + 2T' + \dots)(a(\eta) + a' T + \dots)^2 (1 + 2\tilde{A})$$

$$= a^2(\eta)(1 + 2\mathcal{H}T + 2T' + 2\tilde{A} + \dots)$$

$$\Rightarrow A \to \tilde{A} = A - T' - \mathcal{H}T$$

Similar calculations give

$$B \to B + T - L'$$
$$C \to C - \mathcal{H}T - \frac{1}{3}\nabla^2 L$$
$$E \to E - L$$

We can choose a coordinate system ("gauge") that simplifies the metric perturbations

Newtonian gauge: $B = E = 0$

Convention: $A \to \Psi$, $C \to -\Phi$

$$\boxed{ds^2 = a^2(\eta)\left[(1+2\Psi)d\eta^2 - (1-2\Phi)\delta_{ij}\, dx^i dx^j\right]}$$

$$\Rightarrow g_{\mu\nu} = a^2 \begin{pmatrix} 1+2\Psi & 0 \\ 0 & -(1-2\Phi)\delta_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = a^{-2} \begin{pmatrix} 1-2\Psi & 0 \\ 0 & -(1+2\Phi)\delta_{ij} \end{pmatrix}$$

In the weak-field limit of GR, Ψ plays the role of the gravitational potential (hence "Newtonian")

Comment: Another useful gauge is the spatially-flat gauge $C = E = 0$, for which $g_{ij} = -\delta_{ij}$

10.2 Matter perturbations

$$T^\mu{}_\nu(\eta, \vec{x}) = \bar{T}^\mu{}_\nu(\eta) + \delta T^\mu{}_\nu(\eta, \vec{x})$$

$$= \begin{pmatrix} \bar{\rho} & 0 \\ 0 & -\bar{P}\delta^i_j \end{pmatrix} + \begin{pmatrix} \delta\rho & -(\bar{\rho}+\bar{P})v^i \\ (\bar{\rho}+\bar{P})v^i & -\delta P \delta^i_j - \Pi^i_j \end{pmatrix}$$

v_i : bulk velocity

Π_{ij} : anisotropic stress (traceless tensor)

Ideal fluid: $\Pi_{ij} = 0$ (not valid for neutrinos!)

Definitions: $\delta \equiv \delta\rho/\rho$ (density contrast)

$q^i \equiv (\bar{\rho} + \bar{P})v^i$ (momentum density)

<u>Note</u>: For several fluids $T^{\mu\nu} = \sum_a T^{\mu\nu}_{(a)}$

$\Rightarrow \delta\rho = \sum_a \delta\rho_a \; ; \; \delta P = \sum_a \delta P_a \; ; \; q^i = \sum_a q^i_{(a)}$

As before, write $q_i = \underbrace{\partial_i q}_{\text{scalar}} + \underbrace{\hat{q}_i}_{\text{vector (irrelevant)}}$

$\Rightarrow$ Three indep. scalar perturbations: $\delta, \delta P, q$

Careful: Indices of $T^\mu{}_\nu$ raised and lowered with perturbed metric.

E.g. $\delta T_{00} = a^2(2\Phi\bar{\rho} + \delta\rho)$

Coordinate transformations also change $\delta T^{\mu}{}_{\nu}$:

$$\delta\rho \to \delta\rho - T\bar{\rho}'$$
$$\delta P \to \delta P - T\bar{P}'$$
$$q_i \to q_i - (\bar{\rho} + \bar{P})L_i'$$

However, we can only simplify either $g_{\mu\nu}$ or $T^{\mu}{}_{\nu}$. Choosing Newtonian gauge leaves no freedom to simplify $T^{\mu}{}_{\nu}$.