10.3 Equations of Motion

Calculate $\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda} \right)$

for perturbed metric in Newtonian gauge

Example: $\Gamma^{0}_{00} = \frac{1}{2} g^{0\rho} \left(\partial_0 g_{\rho 0} + \partial_0 g_{0\rho} - \partial_\rho g_{00} \right)$

$$= \frac{1}{2} g^{00} \frac{\partial}{\partial \eta} g_{00}$$

$$= \frac{1}{2} a^{-2} (1 - 2\Psi) \frac{\partial}{\partial \eta} \left[a^2 (1 + 2\Psi) \right]$$

$$= (1 - 2\Psi) \left[\mathcal{H} (1 + 2\Psi) + \Psi' \right]$$

$$= \mathcal{H} + \Psi' + \mathcal{O}(\Psi^2)$$

Similarly: $\Gamma^{0}_{i0} = \partial_i \Psi$

$$\Gamma^{i}_{00} = \partial^i \Psi$$

$$\Gamma^{0}_{ij} = \mathcal{H} \delta_{ij} - \left[\Phi' + 2\mathcal{H}(\Phi + \Psi) \right] \delta_{ij}$$

$$\Gamma^{i}_{j0} = \left[\mathcal{H} - \Phi' \right] \delta^i_j$$

$$\Gamma^{i}_{jk} = -\delta^i_j \partial_k \Phi - \delta^i_k \partial_j \Phi + \delta_{jk} \partial^i \Phi$$

Now consider E-p conservation:

$$0 = \nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\nu\mu} T^\mu{}_\alpha$$

Consider $\nu = 0$ and make use of E-p conservation for background ($\bar{\rho}' = -3\mathcal{H}(\bar{\rho}+\bar{P})$) to find continuity eq.

$$\boxed{\delta\rho' = \underbrace{-3\mathcal{H}(\delta\rho + \delta P)}_{\text{dilution due to expansion}} + \underbrace{3\bar{\Phi}'(\bar{\rho}+\bar{P})}_{\text{relativistic effect}} \underbrace{-\partial_i q^i}_{\text{local fluid flow}}} \quad (C1)$$

For $\nu = i$ find Euler eq.

$$\boxed{q^{i\prime} = \underbrace{-4\mathcal{H}q^i}_{\text{redshift of momenta}} \underbrace{-(\bar{\rho}+\bar{P})\partial^i\Psi - \partial^i\delta P}_{\text{force terms}}} \quad (C2)$$

Note: If there are several (non-interacting) fluids, these eqs. apply to each individually and to their sum

Special cases:

- Matter ($P_m = 0$): $\delta_m' = -\vec{\nabla}\cdot\vec{v}_m + 3\Phi'$

$$\vec{v}_m' = -\mathcal{H}\vec{v}_m - \vec{\nabla}\Psi$$

$$\Rightarrow \delta_m'' + \mathcal{H}\delta_m' = \nabla^2\Psi + 3(\Phi'' + \mathcal{H}\Phi')$$

- Radiation ($P_r = \frac{1}{3} \rho_r$): $\delta_r' = -\frac{4}{3} \vec{\nabla} \cdot \vec{v}_r + 4 \Phi'$

$$\vec{v}_r' = -\frac{1}{4} \vec{\nabla} \delta_r - \vec{\nabla} \Psi$$

$$\Rightarrow \delta_r'' - \frac{1}{3} \nabla^2 \delta_r = \frac{4}{3} \nabla^2 \Psi + 4 \Phi''$$

Every perfect fluid satisfies $\delta P_a = c_{s,a}^2 \delta \rho_a$

adiabatic sound speed

$\Rightarrow$ 2 eqs. for 2 perturbations ($\delta \rho_a$ & v_a) if metric perturbations known

Evolution of metric determined by Einstein eqs.

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2} \delta^\mu_\nu R = 8\pi G T^\mu{}_\nu$$

Background satisfies $\bar{G}^\mu{}_\nu = 8\pi G \bar{T}^\mu{}_\nu$

$$\Rightarrow \delta G^\mu{}_\nu = 8\pi G \, \delta T^\mu{}_\nu$$

Long and tedious calculation: $a^2 \delta G^0{}_0 = 2\nabla^2 \Phi - 6\mathcal{H}(\Phi' + \mathcal{H}\Psi)$

$$\Rightarrow \boxed{\nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 \delta\rho} \quad (E1)$$

total density perturbation $\delta\rho = \sum_a \delta\rho_a$

↳ Relativistic version of Poisson eq.

Space-space part of Einstein eq. gives

I) $\Phi - \Psi = 0$ (for perfect fluid with $\Pi = 0$)

II) $$\boxed{\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P} \qquad (E2)$$

Time-space part of Einstein eq. gives

$$\boxed{\Phi' + \mathcal{H}\Phi = -4\pi G a^2 q} \qquad (E3)$$

Remember: Einstein eq. implies E-p conservation for sum of perturbations

$\Rightarrow$ Not all eqs. independent

↳ For n non-interacting fluids we have $2n+1$ indep. eqs. for $2n+1$ perturb.

For $n=1$ (single fluid) we use $\delta p = c_s^2 \delta\rho$ to get

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = c_s^2\left[\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)\right]$$

$$\Rightarrow \Phi'' + 3\mathcal{H}\Phi'(1 + c_s^2) + \underbrace{\left[2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2)\right]}_{=0 \text{ (Einstein eq. for background)}}\Phi - c_s^2\nabla^2\Phi = 0$$

$$\Rightarrow \Phi'' + 3\mathcal{H}(1+c_s^2)\Phi' - c_s^2 \nabla^2 \Phi = 0$$

Go to Fourier space $(\partial_i \to i k_i)$

$$\Phi'' + 3\mathcal{H}(1+c_s^2)\Phi' + c_s^2 k^2 \Phi = 0$$

$\hookrightarrow$ Each Fourier mode evolves independently

Examples:

1) Matter domination $(c_s^2 = 0)$

$$\Phi'' + 3\mathcal{H}\Phi' = 0 \qquad \text{with } \mathcal{H} = \frac{2}{\eta}$$

Solution a: $\Phi = \text{const} = \Phi_i$

Solution b: $\Phi \sim \eta^{-5} \sim a^{-5/2}$

$\hookrightarrow$ "decaying" solution

$\Rightarrow$ negligible for initial conditions with small perturbations

2) Radiation domination $(c_s^2 = 1/3)$

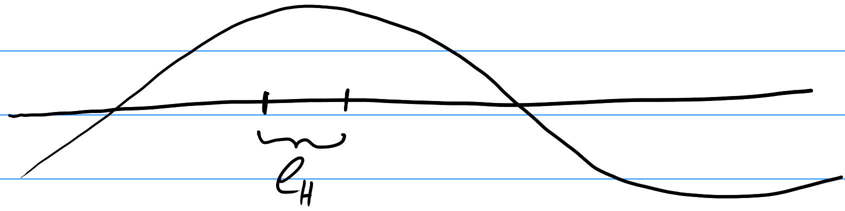
$$\Phi'' + 4\mathcal{H}\Phi' + \frac{1}{3}k^2\Phi = 0 \qquad \text{with } \mathcal{H} = 1/\eta$$

For $k\eta \ll 1$, i.e. $k \ll \mathcal{H}$, neglect final term

$\Rightarrow \Phi = \text{const} \quad (k \ll \mathcal{H})$

$\hookrightarrow$ "superhorizon mode"

Interpretation: Wavelength $\lambda = \frac{2\pi a}{k}$ large compared to particle horizon $\ell_H = 1/H$



$\Rightarrow$ each causally connected region nearly homogeneous

$\Rightarrow$ superhorizon mode "frozen"

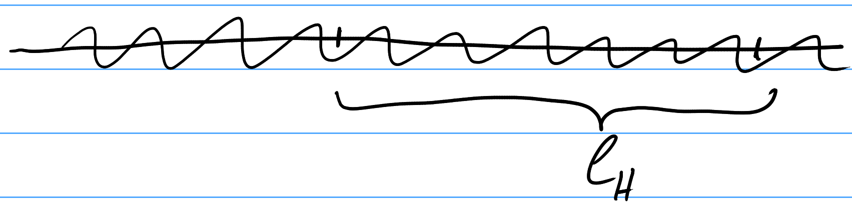
For $k\eta \gg 1$ define $f(\eta) = a^2(\eta)\, \Phi(\eta)$

$$\Rightarrow f'' + \frac{1}{3}k^2 f \underbrace{- 2\mathcal{H}' f - 4\mathcal{H}^2 f}_{\text{can be neglected}} = 0$$

$$\Rightarrow f(\eta) = A \cdot \cos\left(\frac{k\eta}{\sqrt{3}}\right) + B \sin\left(\frac{k\eta}{\sqrt{3}}\right)$$

$$\overset{a \sim \eta}{\Rightarrow} \Phi(\eta) = \frac{A'}{\eta^2} \cos\left(\frac{k\eta}{\sqrt{3}}\right) + \frac{B'}{\eta^2} \sin\left(\frac{k\eta}{\sqrt{3}}\right) \qquad (k \gg \mathcal{H})$$

$\hookrightarrow$ "subhorizon mode" with $\lambda \ll \ell_H$



$\Rightarrow$ oscillations with decreasing amplitude

Note: General solution during RD:

$$\Phi(y) = A\,\frac{j_1(y)}{y} + B\,\frac{n_1(y)}{y} \qquad \left(y \equiv \frac{k\eta}{\sqrt{3}}\right)$$

with
$$j_1(y) = \frac{\sin y}{y^2} - \frac{\cos y}{y}$$
$$n_1(y) = -\frac{\cos y}{y^2} - \frac{\sin y}{y}$$

For $y \to 0$: $\quad j_1(y) \approx \frac{y}{3}, \qquad n_1(y) \approx -\frac{1}{y^2}$

↳ decaying solution

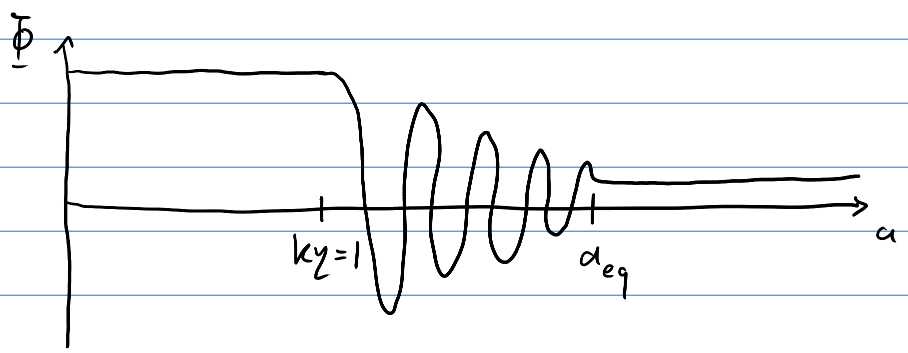
$$\Rightarrow \Phi(\eta) = 3\Phi_i\left(\frac{\sin y - y\cos y}{y^3}\right)$$

Summary: Cosmological evolution of metric perturbations

- In very early Universe, ℓ_H is tiny
  $\Rightarrow$ almost all perturbations outside the horizon
  $\Rightarrow$ no evolution (modes frozen)

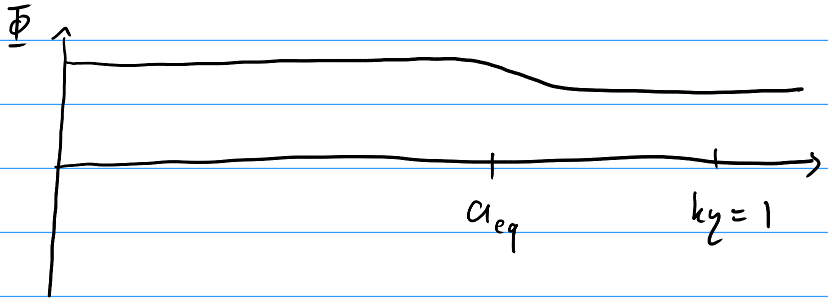
- As Universe expands, more and more modes enter the horizon and start oscillating

- Oscillations stop after M-R equality



$k > k_{eq}$

- Modes that enter the horizon after M-R equality never oscillate



$k < k_{eq}$

$$\lambda_0^{eq} = \frac{2\pi a_0}{k_{eq}} \approx 750\ \mathrm{Mpc}$$

=> Perturbations relevant for the formation of galaxies ($\lambda \sim 10-100$ kpc) enter horizon during RD

<u>Comment:</u> If the Universe contains a single fluid, matter perturbations are completely determined by metric perturbations via (E1)-(E3)

Example: MD ($\Phi = \Phi_i$)

(E1) $\quad k^2 \Phi_i + 3\mathcal{H}^2 \Phi_i = -4\pi G a^2 \delta\rho$

$k\eta \ll 1$: Use $\mathcal{H}^2 = \frac{8\pi}{3} G \bar{\rho} a^2$

$$\Rightarrow \delta = \frac{\delta\rho}{\bar{\rho}} = -2\Phi$$

$k\eta \gg 1$: $\delta\rho \sim a^{-2}$

$$\Rightarrow \delta = \frac{\delta\rho}{\bar{\rho}} \sim a$$

Summary

$$\delta\rho' = -3\mathcal{H}(\delta\rho + \delta P) + 3\Phi'(\bar{\rho} + \bar{P}) - \partial_i q^i \tag{C1}$$

$$q^{i\prime} = -4\mathcal{H} q^i - (\bar{\rho} + \bar{P})\partial^i \Phi - \partial^i \delta P \tag{C2}$$

$$\nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho \tag{E1}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P \tag{E2}$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2 q \tag{E3}$$