10.4 Initial conditions

- Superhorizon modes do not evolve (frozen)
- Need to specify initial conditions at early times (RD)
- For single fluid: Enough to specify Φ_i
- But real Universe contains γ, B, DM, ...

  ↳ Additional assumptions needed

10.4.1 Adiabatic fluctuations

Assume Universe has same composition everywhere:

$$\delta\left(\frac{n_B}{s}\right) = \delta\left(\frac{n_{DM}}{s}\right) = \dots = 0$$

But some parts are hotter than others: $\delta T \neq 0$

Interpretation: Some parts are "ahead" and some are "behind": $\eta \to \eta + \delta\eta(\vec{x})$

$$\Rightarrow \delta T(\eta, \vec{x}) = T(\eta + \delta\eta(\vec{x})) - T(\eta)$$

$$= T' \, \delta\eta(\vec{x})$$

$\Rightarrow$ All perturbations proportional to $\delta\eta(\vec{x})$

$$\frac{\delta\rho_a(\vec{x})}{\bar{\rho}_a'} = \frac{\delta\rho_b(\vec{x})}{\bar{\rho}_b'}$$

$$\bar{\rho}_a' = -3\mathcal{H}(1+w_a)\bar{\rho}_a \Rightarrow \frac{\delta_a}{1+w_a} = \frac{\delta_b}{1+w_b}$$

for all species a, b

<u>Note</u>: Perturbations with $\frac{\delta_a}{1+w_a} \neq \frac{\delta_b}{1+w_b}$ but $\delta T = 0$

are called <u>isocurvature perturbations</u>

↳ no observational evidence

↳ contribution must be tiny

Adiabatic perturbations: $\underset{\substack{\uparrow \\ w \approx 0}}{\delta_B} = \underset{\substack{\uparrow \\ w \approx 0}}{\delta_{DM}} = \frac{3}{4}\underset{\substack{\uparrow \\ w = 1/3}}{\delta_\gamma}$

$\Rightarrow \delta\rho = \sum_a \bar{\rho}_a \delta_a$ dominated by species with largest $\bar{\rho}_a$

(E1) $\quad k^2\Phi_i + 3\mathcal{H}^2\Phi_i = -4\pi G a^2 \delta\rho$

$k\eta \ll 1$: Use $\mathcal{H}^2 = \frac{8\pi}{3} G \bar{\rho} a^2$

$$\Rightarrow \delta = \frac{\delta\rho}{\bar{\rho}} = -2\Phi$$

RD: $\delta_r = \frac{4}{3}\delta_{DM} = -2\Phi_i$ (superhorizon)

$\Rightarrow$ Initial conditions fully specified by Φ_i

MD: $\delta \approx \delta_{DM} = -2\Phi = \text{const}$ (superhorizon)

$\Rightarrow$ Φ cannot stay const around MR equality

↳ Need alternative variable

↳ Consider $\mathcal{R} \equiv -\Phi + \frac{\mathcal{H}}{\bar{\rho} + \bar{P}} q$

"curvature perturbation"   "momentum potential"

$q_i = \partial_i q$

For $k \ll \mathcal{H}$: (E1) + $3\mathcal{H}$(E3): $\mathcal{H} q = \frac{1}{3}\delta\rho$ (*)

$$\Rightarrow \mathcal{R} = -\Phi + \frac{\delta\rho}{3(\bar{\rho} + \bar{P})}$$

$$\Rightarrow \frac{\partial \mathcal{R}}{\partial \eta} = -\frac{\partial \Phi}{\partial \eta} + \frac{\delta\rho'}{3(\bar{\rho} + \bar{P})} - \frac{\delta\rho}{3(\bar{\rho} + \bar{P})^2}(\bar{\rho}' + \bar{P}')$$

(C1): $\delta\rho' = -3\mathcal{H}(\delta\rho + \delta P) + 3\Phi'(\bar{\rho} + \bar{P})$

$$\Rightarrow \frac{\partial \mathcal{R}}{\partial \eta} = -\mathcal{H}\frac{\delta\rho + \delta P}{(\bar{\rho} + \bar{P})} - \frac{\delta\rho}{3(\bar{\rho} + \bar{P})^2}(\bar{\rho}' + \bar{P}')$$

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho}+\bar{P})$$

$$= -\mathcal{H}\frac{\delta P}{(\bar{\rho}+\bar{P})} + \mathcal{H}\frac{\delta\rho\,\bar{P}'}{(\bar{\rho}+\bar{P})\,\bar{\rho}'}$$

$$= \frac{\mathcal{H}\,\bar{P}'}{\bar{\rho}+\bar{P}}\underbrace{\left(\frac{\delta\rho}{\bar{\rho}'} - \frac{\delta P}{\bar{P}'}\right)}_{=0 \text{ for adiabatic perturbations}}$$

$\Rightarrow$ $\mathcal{R}' = 0$ for superhorizon modes

↳ conservation of curvature perturbations

Note: For $\bar{\rho}+\bar{P} = (1+w)\,\bar{\rho}$

$$\mathcal{R} \overset{(*)}{=} -\Phi + \frac{\delta}{3(1+w)}$$

$$= -\Phi - \frac{2\Phi}{3(1+w)} = -\frac{5+3w}{3+3w}\,\Phi$$

RD: $\mathcal{R}_{RD} = -\frac{3}{2}\Phi_{RD}$

MD: $\mathcal{R}_{MD} = -\frac{5}{3}\Phi_{MD}$

$\mathcal{R}_{RD} = \mathcal{R}_{MD} \Rightarrow \Phi_{MD} = \frac{9}{10}\Phi_{RD}$

↳ Metric perturbations decrease slightly at MR equality for superhorizon modes

$\Rightarrow \delta_\gamma = \frac{4}{3}\delta_{DM} = -\frac{8}{3}\Phi_{MD} = -\frac{12}{5}\Phi_{RD}$

↳ Matter perturb. increase slightly

10.4.2 Gaussian random fields

$\mathcal{R}(\vec{x})$ or $\mathcal{R}(\vec{k})$ fully specifies adiabatic perturbations for modes that enter horizon at any time during RD or MD

What is $\mathcal{R}(\vec{x})$?

Not possible to predict $\mathcal{R}(\vec{x})$ exactly, but possible to predict statistical properties

Simplest case: For all $\vec{x}$ $\mathcal{R}(\vec{x})$ follows normal dist. with

$$\langle \mathcal{R}(\vec{x}) \rangle = 0$$

$$\langle \mathcal{R}(\vec{x})^2 \rangle = \sigma^2$$

indep. of $\vec{x}$

$\Rightarrow$ Need to specify correlation between nearby points

$\hookrightarrow$ Correlation function $\langle \mathcal{R}(\vec{x}) \mathcal{R}(\vec{x})' \rangle \equiv \xi_{\mathcal{R}}(\vec{x}, \vec{x}')$

Translation and rotation invariance: $\xi_{\mathcal{R}}(\vec{x}, \vec{x}') = \xi_{\mathcal{R}}(|\vec{x} - \vec{x}'|)$
(generalized cosmological principle)

Fourier space:

$$\langle \mathcal{R}(\vec{k}) \mathcal{R}^*(\vec{k}') \rangle = \frac{1}{(2\pi)^6} \int d^3x \, d^3y \, e^{-i\vec{k}\vec{x} + i\vec{k}\vec{y}} \xi_{\mathcal{R}}(|\vec{x} - \vec{y}|)$$

$$= \frac{1}{(2\pi)^6} \int d^3x \, e^{-i\vec{x}(\vec{k} - \vec{k}')} \int d^3z \, e^{i\vec{k}'\vec{z}} \xi_{\mathcal{R}}(z)$$

$$= \delta^3(\vec{k}-\vec{k}')\,\frac{P(k)}{(2\pi)^3}$$

$$\text{with} \quad P(k) \equiv \int d^3z \; e^{i\vec{k}\vec{z}}\, \xi_{\mathcal{R}}(z)$$

Convenient to define $\Delta^2_{\mathcal{R}}(k) \equiv \frac{k^3 P(k)}{2\pi^2}$ (dimensionless)

$$\Rightarrow \quad \boxed{\langle \mathcal{R}(\vec{k})\,\mathcal{R}^*(\vec{k}')\rangle = \frac{1}{4\pi k^3}\,\Delta^2_{\mathcal{R}}(k)\,\delta^3(\vec{k}-\vec{k}')}$$

↳ $\Delta_{\mathcal{R}}$ fully characterises Gaussian random field $\mathcal{R}(\vec{k})$ and hence the statistical properties of initial perturbations

↳ no observational evidence for (and strong bounds on) non-Gaussianity

Simplest assumption: Scale-invariant spectrum (Harrison-Zeldovich spectrum)

$$\Delta^2_{\mathcal{R}}(k) = \text{const}$$

Observations: Small deviation from scale invariance

$$\Delta^2_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s-1}$$

with $A_s \approx 2.1 \cdot 10^{-9}$ (for $k_* = 0.05\ \mathrm{Mpc}^{-1}$)

↑ matter power amplitude

$n_s = 0.9665 \pm 0.0038$

↑ spectral index

No evidence for more complicated scaling $\left(\frac{dn_s}{d\log k} \approx 0\right)$

$\Rightarrow$ Initial perturbation at level $\Delta_R \sim 5 \cdot 10^{-5}$

Consider again mode that enters horizon after MR equality $(a_\times > a_{eq})$

crossing scale: $k \cdot \eta = 1$

For $k \cdot \eta \gg 1$: $\quad k^2 \Phi \overset{(E1)}{\approx} -4\pi G a^2 \delta\rho$

$$\Rightarrow \delta\rho \sim a^{-2} \quad \Rightarrow \quad \delta = \frac{\delta\rho}{\bar\rho} \sim a$$

$$\delta(a > a_\times) \approx \delta_i \cdot \frac{a}{a_\times}$$

$$= \frac{9}{5}\Phi_i \cdot \frac{a}{a_\times}$$

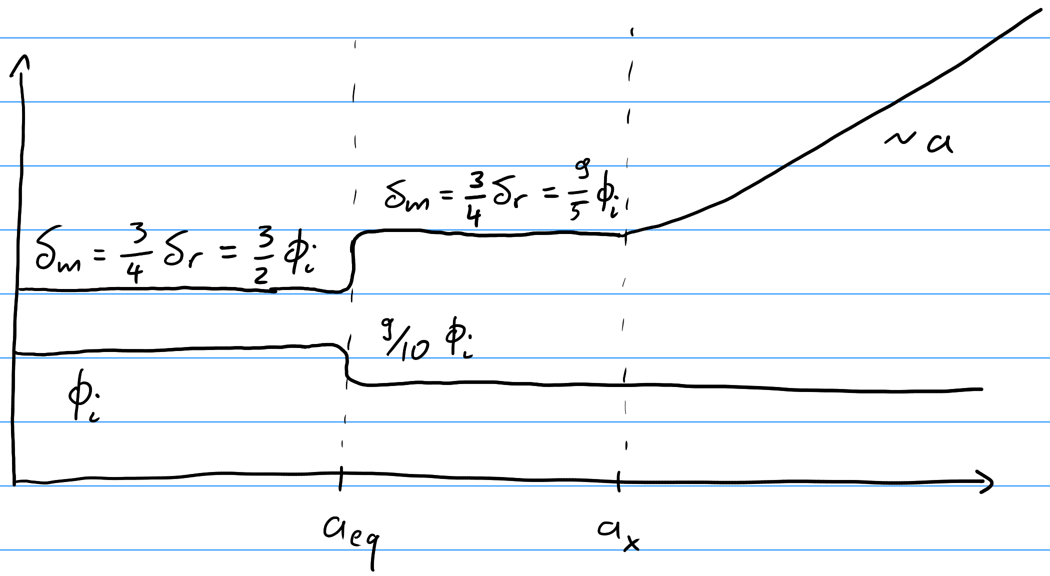
$$\sim 10^{-4} \frac{a}{a_\times} < 10^{-4} \frac{a}{a_{eq}}$$

$\frac{a_0}{a_{eq}} = 3.2 \cdot 10^3 \quad \Rightarrow \quad \delta(a_0) \lesssim 0.3$

↳ modes that enter during MD are still in linear regime today

$\Rightarrow$ Universe homogeneous at scales $> 100\,\mathrm{Mpc}$

$\Rightarrow$ No gravitationally bound structures $> 10^{16} M_\odot$



What about smaller scales? → Next lecture!