§11 Structure Formation

How do tiny initial perturbations grow into macroscopic structures?

→ Gravitational instability

↳ Overdense regions attract more matter

- ⇒ overdensity grows
- ⇒ temperature increases
- ⇒ pressure increases
- ⇒ matter pushed outwards
- ⇒ growth of structure inhibited

Which effect wins?

Consider fluid with $\bar{P} = w\bar{\rho}$ & $\delta P = c_s^2 \delta\rho$

(typically $w = c_s^2$)

(C1) $$\delta' = -(1+w)(\vec{\nabla}\cdot\vec{v} - 3\Phi') + 3\mathcal{H}\underbrace{(w - c_s^2)}_{\approx 0}$$

$$\overset{\text{subhorizon}}{\approx} -(1+w)\,\vec{\nabla}\vec{v}$$

(C2) $$\vec{v}\,' = -\mathcal{H}(1-3w)\vec{v} - \frac{c_s^2}{1+w}\vec{\nabla}\delta - \vec{\nabla}\Phi$$

$$\Rightarrow \delta'' = \mathcal{H}(1-3w)\underbrace{(1+w)\vec{\nabla}\vec{v}}_{=-\delta'} + \underbrace{c_s^2\nabla^2\delta}_{=-c_s^2 k^2\delta} + \underbrace{(1+w)\nabla^2\Phi}_{=(1+w)4\pi G a^2\bar{\rho}}$$

(subhorizon)

Define $\quad k_J^2 \equiv (1+w)\dfrac{4\pi G a^2 \bar{\rho}}{c_s^2}$ (Jeans length)

$$\Rightarrow \quad \boxed{\delta'' + \underbrace{(1-3w)\mathcal{H}\delta'}_{\text{friction}} + c_s^2(\underbrace{k^2}_{\text{pressure}} - \underbrace{k_J^2}_{\text{gravity}})\delta = 0} \quad (*)$$

For $\quad k > k_J$: Pressure dominates

$\Rightarrow$ fluctuations oscillate

$k < k_J$: Gravity dominates

$\Rightarrow$ fluctuations grow

↳ Jeans instability

For DM: $c_s^2 \approx 0 \Rightarrow$ fluctuations grow on all scales

For γ: $w \approx c_s^2 \approx \frac{1}{3} \Rightarrow$ acoustic (i.e. pressure) oscillations

Recap: Matter perturbations grow linearly during MD

<u>But</u>: Not enough time to form non-linear structures

$\Rightarrow$ Visible structures correspond to modes entering horizon during RD

<u>But</u>: Radiation perturbations do not grow during RD

$\Rightarrow$ Need to consider matter & radiation at once!

11.1 Clustering of Dark Matter

Ignoring baryons (i.e. $\Omega_m = \Omega_{DM}$) we can write

$$\mathcal{H}^2 = \mathcal{H}_0^2 \left(\Omega_m a^{-1} + \Omega_r a^{-2} \right) = \frac{\mathcal{H}_0^2 \Omega_m^2}{\Omega_r} \left(\frac{1}{y} + \frac{1}{y^2} \right)$$

$\uparrow\ y = a/a_{eq}$

Note: Perturbations in radiation oscillate rapidly
⇒ Average out on Hubble timescale
⇒ Metric perturbations sourced by DM even during RD: $\nabla^2 \Phi = 4\pi a^2 G \bar{\rho}_m \delta_m$

⇒ Can use result from above with $w \approx c_s^2 \approx 0$

(*)

$$\Rightarrow \quad \delta_m'' + \mathcal{H} \delta_m' - 4\pi G a^2 \bar{\rho}_m \delta_m \approx 0$$

$$y' = \frac{a'}{a_{eq}} = \mathcal{H} y \quad \Rightarrow \quad \frac{\partial}{\partial \eta} = \mathcal{H} y \frac{\partial}{\partial y}$$

$$\Rightarrow \mathcal{H} \delta_m' = \frac{\mathcal{H}_0^2 \Omega_m^2}{\Omega_r} \left(1 + \frac{1}{y} \right) \frac{\partial \delta_m}{\partial y}$$

$$\delta_m'' = \frac{\mathcal{H}_0^2 \Omega_m^2}{\Omega_r} \left(\frac{1}{2} \frac{\partial \delta_m}{\partial y} + (1+y) \frac{\partial^2 \delta_m}{\partial y^2} \right)$$

Using $\frac{8\pi}{3} G a^2 \bar{\rho}_{m} = \frac{\mathcal{H}_0^2 \Omega_m^2}{\Omega_r} \frac{1}{y}$ we obtain

$$\boxed{\frac{\partial^2 \delta_m}{\partial y^2} + \frac{2+3y}{2y(1+y)}\frac{\partial \delta_m}{\partial y} - \frac{3}{2y(1+y)}\delta_m = 0}$$

(Mészáros eq.)

Ansatz: $\delta_m = (a + by) f(y)$

Solutions: $\delta_m^{(1)} = c_1 \left(1 + \frac{3}{2}y\right)$

$$\delta_m^{(2)} = c_2 \left(1 + \frac{3}{2}y\right)\left[\log \frac{\sqrt{1+y}-1}{\sqrt{1+y}+1} + 6\frac{\sqrt{1+y}}{2+3y}\right]$$

For $y \ll 1$ (RD): $\delta_m = c_1 + c_2 \log y$

$$= c_1 + c_2 \log y/y_{eq}$$

Matching to initial conditions (lengthy calculation):

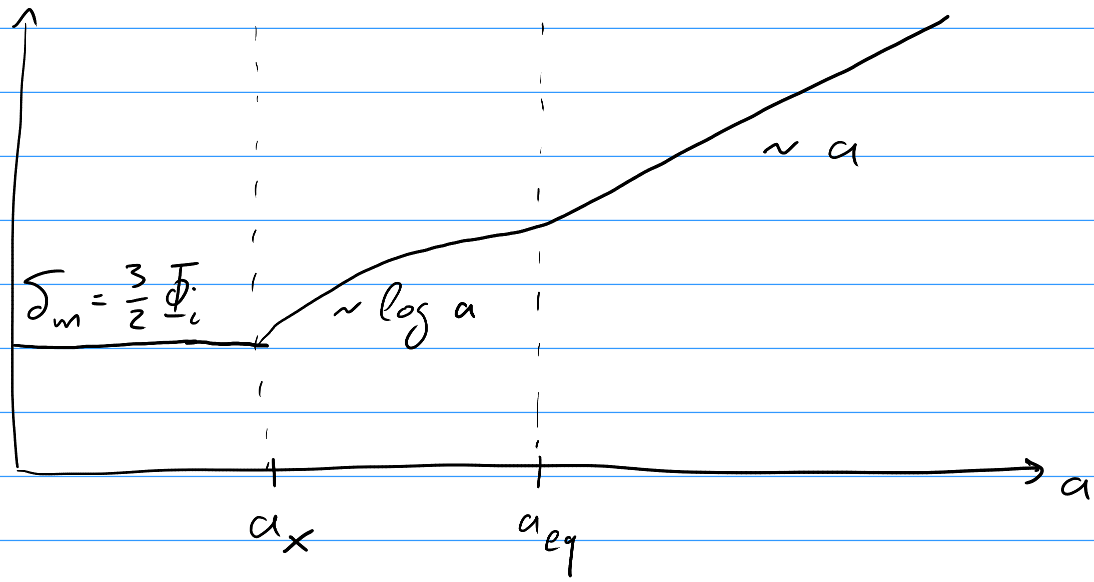
$c_2 = -9\Phi_i$ $\qquad$ $c_1 = -9\Phi_i \log k\eta_{eq}$

$\Rightarrow \delta_m = -9\Phi_i \log k\eta$

For $y \gg 1$ (MD): $\delta_m = \frac{3}{2}c_1 y$

$$= -\frac{27}{2}\Phi_i \log(k\eta_{eq})\frac{a}{a_{eq}}$$

$\sim a$ (as expected)



Notes: - In contrast to radiation, dark matter perturbations do not oscillate during RD

Reason: No pressure $\leftrightarrow$ Jeans instability at all scales

- Matter perturbations significantly enhanced already in RD compared to $\delta_{m,i} = -\frac{3}{2}\Phi_i$

- When including baryons, we have

$$\frac{8\pi}{3} G a^2 \bar{\rho}_{DM} = \frac{\Omega_{DM}}{\Omega_m} \frac{H_0^2 \Omega_m^2}{\Omega_r} \frac{1}{y}$$

$\Rightarrow$ Final term in Mészáros eq. becomes

$$-\frac{3}{2y(1+y)} \frac{\Omega_{DM}}{\Omega_m} \delta_m$$

$\Rightarrow$ Suppression of $\delta_{DM} \sim \left(1 - \frac{0.6\,\Omega_b}{\Omega_m}\right)$

11.2 Matter Power Spectrum

↳ Characterises amount of structure in late universe (during MD)

$$\langle \delta_m(\vec{k})\, \delta_m^*(\vec{k}') \rangle = \frac{P_m(k)}{(2\pi)^3}\, \delta^3(\vec{k} - \vec{k}')$$

Introduce transfer function: $\delta_m(k,a) = T_m(k,a)\mathcal{R}(k)$

$$\Rightarrow \quad P_m(k,a) = T_m(k,a)^2\, P(k)$$

$$= \frac{2\pi^2}{k^3}\, T_m(k,a)^2\, \underbrace{\Delta_{\mathcal{R}}^2(k)}_{\approx\ \text{const}}$$

For $k < k_{eq}$ (horizon entry after MR equality)

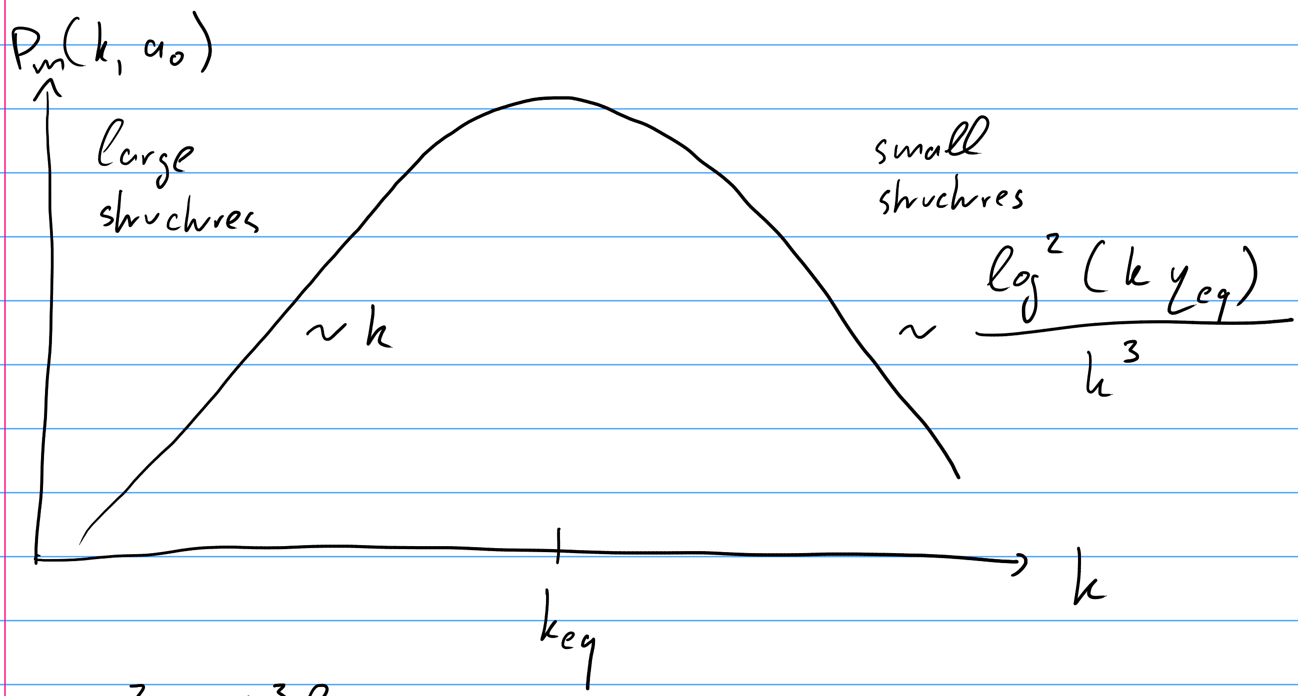
$$\delta_m \sim \frac{a}{a_x}\Phi_i \overset{MD}{\sim} \frac{\eta^2}{\eta_x^2}\Phi_i \overset{\eta_x k = 1}{\sim} k^2 \eta^2 \Phi_i$$

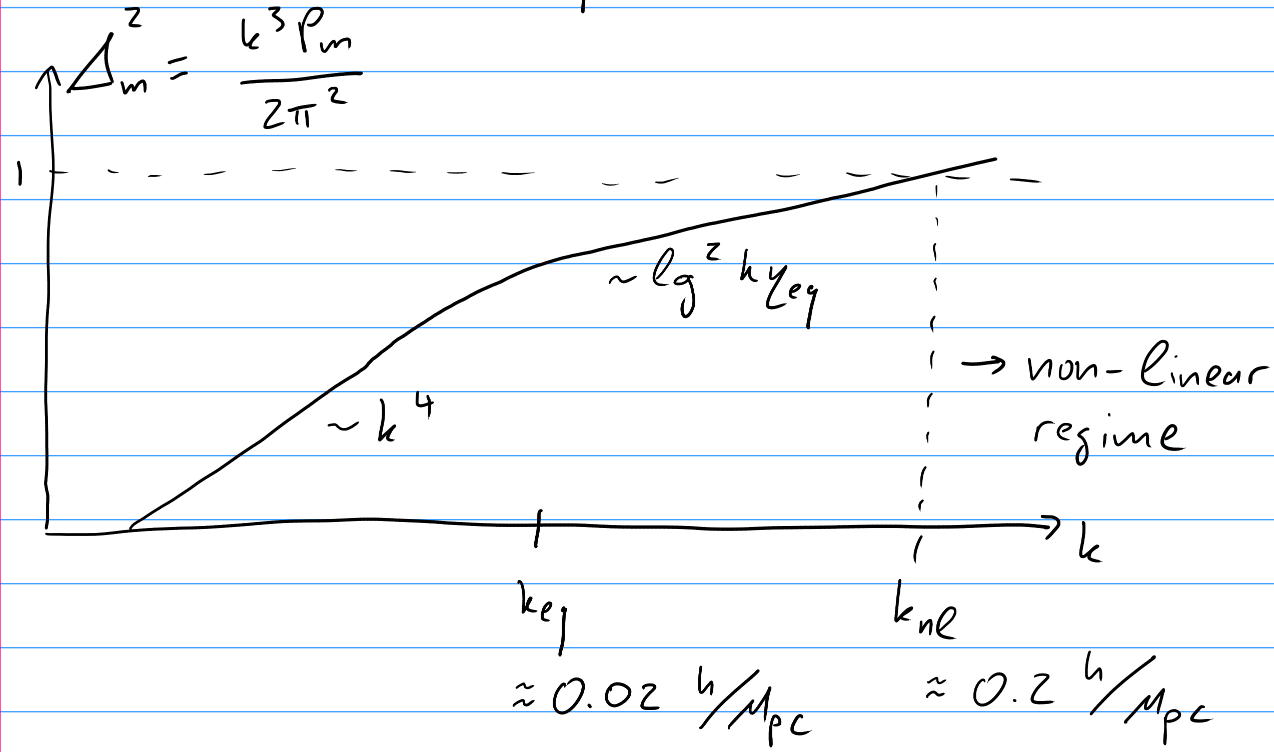
$$\Rightarrow T_m(k,a) \sim k^2 a \quad \Rightarrow \quad P_m(k,a) \sim k \cdot a^2$$

For $k > k_{eq}$ (horizon entry before MR equality)

$$\delta_m \sim \log(k\,\eta_{eq})\, a\, \Phi_i$$

$$P_m(k,a) \sim \frac{\log^2(k\eta_{eq})}{k^3} a^2$$





Including baryons: - Slight suppression of $P_m(k)$
- Baryon acoustic oscillations
↳ next lecture