§13 Inflation

13.1 Motivation

1) Initial conditions

↳ In ΛCDM the initial conditions (A_s, n_s) are introduced by hand

↳ Would like to predict them from more fundamental theory

2) Horizon problem

↳ Particle horizon at decoupling:

$$r_{dec} = \eta_{dec} = \frac{2}{\mathcal{H}_{dec}}$$

$$\Delta\vartheta_H = \frac{r_{dec}}{\eta_0} = (1+z_{dec})^{-1/2} \approx 1.7°$$

$\Rightarrow$ There should be no correlations in the CMB for $\vartheta > \Delta\vartheta_H$ $(\ell < 100)$

Why is the CMB temperature (approximately) the same everywhere?

3) Flatness problem

Remember: $H^2 = \frac{8\pi}{3 M_p^2}(\rho_m + \rho_{rad} + \rho_\Lambda) - \frac{k}{a^2}$

↳ CMB measurements give

$$|\Omega_{curv}| = \frac{|k|}{a^2 H_0^2} \lesssim 10^{-2}$$

Define $\rho_{curv}^{eff} = \frac{3 k M_p^2}{8\pi a^2}$

$$\Rightarrow \frac{\rho_{curv,0}^{eff}}{\rho_{rad,0}} = \frac{\Omega_{curv}}{\Omega_{rad}} \lesssim 100$$

At earlier times: $\rho_{curv}^{eff} \sim a^{-2}$
$\rho_{rad} \sim a^{-4}$

$$\Rightarrow \frac{\rho_{curv}^{eff}}{\rho_{rad}} \sim a^2 \overset{RD}{\sim} t$$

$\Rightarrow$ Importance of curvature grows with time

$\Rightarrow$ Curvature must have been tiny in the early universe

Example: Consider Planck time $t_p = \frac{1}{M_{pl}} \sim 10^{-43}\,s$

$$\frac{\rho_{curv}^{eff}(t=t_p)}{\rho_{rad}(t=t_p)} = \frac{t_p}{t_0}\frac{\Omega_{curv}}{\Omega_{rad}} = \frac{10^{-43}\,s}{10^{20}\,s}\frac{\Omega_{curv}}{\Omega_{rad}} \lesssim 10^{-60}$$

To avoid extrem fine-tuning, need a mechanism to dilute ρ_{curv}^{eff}

Key observation: $$\frac{8\pi}{3H^2 M_p^2}(\rho_M + \rho_{rad} + \rho_\Lambda) = 1 + \frac{k}{H^2 a^2}$$

$\Rightarrow$ Importance of curvature decreases if $H \cdot a = \dot{a}$ increases

$\Leftrightarrow$ accelerated expansion

Remember: For vacuum energy domination

$$a \sim e^{Ht} \quad \text{with} \quad H = \sqrt{\frac{8\pi}{3M_p^2}\rho_\Lambda}$$

$\Rightarrow$ Flatness problem solved by early period of accelerated expansion

$\hookrightarrow$ Inflation

Example: Assume inflation takes place between t_{ini} and t_{end}

$$\frac{\rho_k^{eff}(t_{end})}{\rho_\Lambda(t_{end})} = \frac{\rho_k^{eff}(t_{ini})}{\rho_\Lambda(t_{ini})} \cdot \left(\frac{a(t_{end})}{a(t_{ini})}\right)^{-2}$$

Assume ρ_Λ is converted into ρ_{rad} at $t = t_{end}$ (reheating)

$$\Rightarrow \frac{\rho_k^{eff}(t_{end})}{\rho_{rad}(t_{end})} \ll 1 \quad \text{if} \quad a(t_{end}) \gg a(t_{ini})$$

Amount of inflation quantified by $e^{\Delta N} = \frac{a(t_{end})}{a(t_{ini})}$

ΔN: Number of e-folds

Flatness problem solved for $\Delta N \gtrsim 60$

Back to problem 2)

A photon emitted at t_1 travels the physical distance

$$\ell_H(t_2, t_1) = a(t_2) \int_{t_1}^{t_2} \frac{dt}{a(t)} = a_2 \int_{a_1}^{a_2} \frac{da}{a^2 H} \overset{\Lambda D}{=} \frac{a_2}{H}\left(\frac{1}{a_1} - \frac{1}{a_2}\right)$$

$$\approx \frac{1}{H} \frac{a_2}{a_1}$$

For $a_1 \to 0$, ℓ_H diverges

=> Photons can travel arbitrarily large distance

=> Entire observable universe becomes causally connected!

↳ Horizon problem solved!

Back to problem 1)

During ΛD $\mathcal{H} = aH = \dot{a}$ increases

=> sub-horizon modes $(k > \mathcal{H})$ become super-horizon

$\Rightarrow$ Perturbations on super-horizon scales can be generated from sub-horizon perturbations that leave the horizon

$\hookrightarrow$ Microscopic fluctuations stretched to macroscopic scales

Comment: - Not necessary to have $H = \text{const}$ as long as $\ddot{a} > 0$

$$\Rightarrow 0 < \frac{\ddot{a}}{(\dot{a})^2} = -\frac{d}{dt}(\dot{a}^{-1}) = -\frac{d}{dt}(aH)^{-1}$$
$$= \frac{\dot{a}H + a\dot{H}}{(aH)^2}$$

$$\Rightarrow 0 < H^2 + \dot{H} = (1 - \varepsilon)$$
$$\hookrightarrow \varepsilon \equiv -\frac{\dot{H}}{H^2}$$

$$\Rightarrow \varepsilon < 1$$

- In order to achieve sufficiently long inflation, we also require

$$\eta \equiv \frac{\dot{\varepsilon}}{H\varepsilon} < 1$$

$\Rightarrow$ Slow-roll inflation

13.2 Inflation from scalar fields

Consider scalar field $\phi(t,\vec{x})$ ("inflaton")

Potential energy given by $V(\phi)$ (e.g. $V(\phi)=\frac{1}{2}m^2\phi^2$)

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi) \underset{\phi(t,\vec{x})=\phi(t)}{=} \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$\Rightarrow$ Eq. of motion: $\boxed{\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0}$

↳ Klein-Gordon eq. with friction term $\sim H\dot{\phi}$

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$$

$$\Rightarrow \left.\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ P_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned}\right\} \Rightarrow \rho_\phi = -P_\phi$$

if kinetic energy negligible

$\Rightarrow$ Friedmann eq.: $\boxed{H^2 = \frac{8\pi}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)}$

$$\Rightarrow 2H\dot{H} = \frac{8\pi}{3M_p^2}\left(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\right)$$

$$V'(\phi) = -\ddot{\phi} - 3H\dot{\phi}$$

$$= -\frac{8\pi H}{M_p^2}\dot{\phi}^2 \quad \Rightarrow \quad \dot{H} = -4\pi\frac{\dot{\phi}^2}{M_p^2}$$

$$\Rightarrow \quad \varepsilon = \frac{4\pi \dot{\phi}^2}{M_p^2 H^2} = \frac{3}{2} \frac{\dot{\phi}^2}{\rho_\phi}$$

↳ Slow-roll inflation requires that kinetic energy makes small contribution to total energy density

Assume $\varepsilon \ll 1$:

$$H^2 \approx \frac{8\pi V}{3 M_p^2}$$

$$3H\dot{\phi} \approx -V'(\phi)$$

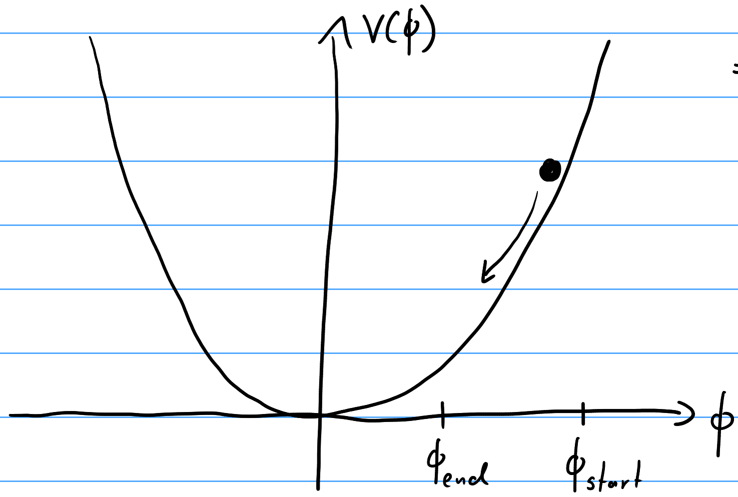
$$\Rightarrow \varepsilon \approx \frac{M_p^2}{16\pi} \left(\frac{V'}{V}\right)^2$$

$$\eta = \frac{\dot{\varepsilon}}{H\varepsilon} = \ldots = \frac{M_p^2}{4\pi} \frac{V''}{V} - 4\varepsilon$$

$$\equiv 2\eta_V - 4\varepsilon_V$$

with $\varepsilon_V = \frac{M_p^2}{16\pi}\left(\frac{V'}{V}\right)^2$, $\eta_V = \frac{M_p^2}{8\pi} \frac{V''}{V}$

Example: $V(\phi) = \frac{1}{2} m^2 \phi^2 \quad \Rightarrow \quad \varepsilon_V = \eta_V = \frac{M_p^2}{4\pi \phi^2}$



$\Rightarrow$ Slow-roll for $\phi \gtrsim M_p$

Inflation ends when $\varepsilon_V \approx 1$

$$\Rightarrow \phi_{end} = \frac{M_p}{\sqrt{4\pi}}$$

$$\Delta N = \ln \frac{a_{end}}{a_{start}} = \int_{a_{start}}^{a_{end}} \frac{da}{a} = \int_{t_{start}}^{t_{end}} H \, dt$$

$$= \int_{\phi_{start}}^{\phi_{end}} \frac{H}{\dot{\phi}} \, d\phi$$

$$= -\frac{2}{M_p} \int_{\phi_{start}}^{\phi_{end}} \sqrt{\frac{\pi}{\varepsilon_V}} \, d\phi$$

$$= -\frac{4\pi}{M_p^2} \int_{\phi_{start}}^{\phi_{end}} \phi \, d\phi$$

$$= \frac{2\pi}{M_p^2} \left(\phi_{start}^2 - \frac{M_p^2}{4\pi} \right)$$

$$= \frac{2\pi \, \phi_{start}^2}{M_p^2} - \frac{1}{2}$$

$\Rightarrow \Delta N > 60$ for $\phi_{start} \gtrsim 3 M_p$