Introduction to Theoretical Particle Physics
Exercise Sheet 2
Winter term 2023/24
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Exercise 2.1: Noether Current **8P**

The dynamics of a complex scalar field $\varphi(x)$ is governed by the Lagrangian

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi - \frac{\lambda}{2} \left(\varphi^* \varphi\right)^2 , \tag{1.1}$$

where $\lambda \in \mathbb{R}$.

(a) Write down the Euler–Lagrange field equations for this system.

(b) Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta\varphi = \mathrm{i}\alpha\varphi, \qquad \delta\varphi^* = -\mathrm{i}\alpha\varphi^*, \tag{1.2}$$

with $\alpha \in \mathbb{R}$.

(c) Derive the Noether current associated with this transformation.

(d) Verify explicitly that the Noether current is conserved using the field equation satisfied by φ found above.

Exercise 2.2: Electromagnetism **12P**

Consider the Lagrangian for electromagnetism:

$$\mathcal{L}_{\mathrm{EM}}(A, \partial A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor and $J^\mu = (\rho, \vec{J})$ is a source.

(a) Show that the Euler-Lagrange equations are $\partial_\mu F^{\mu\nu} = J^\nu$. In order to perform the variation, treat each component of A_μ as an independent scalar field.

(b) Rewrite the Euler-Lagrange equations in terms of the electric and magnetic fields: $E^i = -F^{0i}$ and $B^i = -(1/2)\varepsilon^{ijk}F^{jk}$. Show that by doing this, you recover two of Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \rho \ , \qquad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{J} \ . \tag{2.2}$$

Hint: Remember that $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

(c) Following Fermi, the Lagrangian can also be formulated as

$$\mathcal{L}_{\mathrm{EM,\ Fermi}}(A, \partial A) = -\frac{1}{2} (\partial_\mu A_\nu)(\partial^\mu A^\nu) - J^\mu A_\mu \ . \tag{2.3}$$

Which necessary condition is required to reproduce the standard form of Maxwell's equations you derived before?

Let us consider source-free electromagnetism for simplicity in the following.
The energy-momentum tensor $T^{\mu\nu}$ is given by

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\rho)}\partial^\nu A_\rho - g^{\mu\nu}\mathcal{L} = -F^{\mu\rho}\partial^\nu A_\rho + g^{\mu\nu}\left(\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right). \tag{2.4}$$

It is conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$. Note that it is not a symmetric tensor, $T^{\mu\nu} \neq T^{\nu\mu}$. Define now a new energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{2.5}$$

where the tensor K is antisymmetric on its first two indices: $K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}$.

(d) Show that $\hat{T}^{\mu\nu}$ is also conserved ($\partial_\mu \hat{T}^{\mu\nu} = 0$). Show furthermore that for the choice $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$, $\hat{T}^{\mu\nu}$ is a symmetric tensor (you can use the Euler-Lagrange equations to do so). Are the tensors $T^{\mu\nu}$ and $\hat{T}^{\mu\nu}$ invariant under gauge transformations

$$A^\mu \to A^\mu + \partial^\mu\lambda(x)\,? \tag{2.6}$$

(e) For the choice of $\hat{T}^{\mu\nu}$ in the previous subquestion, show that the energy and momentum of the field are given by the familiar expressions:

$$\begin{aligned} P^0 &= \int \mathrm{d}^3\vec{x}\ \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2) \\ \vec{S} &= \int \mathrm{d}^3\vec{x}\ (\vec{E}\times\vec{B}). \end{aligned} \tag{2.7}$$