Exercise 3.1: SO(N) and SU(N) **10P**

Consider the Lie group SO(N) of orthogonal (i.e. $M^{\mathrm{T}}M = \mathbb{1}$) matrices with unit determinant and the Lie group SU(N) of unitary (i.e. $M^\dagger M = \mathbb{1}$) matrices with unit determinant.[1]

(a) Determine the dimension (i.e. the number of different matrices whose *real* linear combinations span the whole group) of SO(N) and of SU(N) for a generic $N \in \mathbb{N}$.

(b) Derive the conditions fullfilled by the corresponding Lie algebras $\mathfrak{so}(N)$ and $\mathfrak{su}(N)$. Remember that any element of a Lie group around the identity can be obtained through the exponentiation of an element of the corresponding Lie algebra: $G \ni M(\alpha) = \exp(\alpha N)$, $N \in \mathfrak{g}$.

Hint: *Remember that* $\det A = \exp \mathrm{Tr} A$.

Focus now on the $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ Lie algebras. In the *fundamental representation*, a basis for $\mathfrak{su}(2)$ is given by $T_F^a = \sigma^a/2$, where σ^a are the Pauli matrices, while a basis for $\mathfrak{so}(3)$ is given by $(T_F^i)_{jk} = \epsilon_{ikj}$, where ϵ_{ijk} is the three-dimensional Levi-Civita symbol, with $\epsilon_{123} = +1$.

(c) Calculate all the structure constants ${f^{ab}}_c$ for both Lie algebras.

(d) The *Cartan–Killing form* of a Lie algebra is defined as

$$K^{ab} = -{f^{am}}_l {f^{bl}}_m \tag{1.1}$$

Prove that the Cartan–Killing form is symmetric. Calculate it explicitely for $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$.

The *adjoint representation* of a Lie algebra is given by the structure constants themselves:

$${(T_A^a)^b}_c = -\mathrm{i}{f^{ab}}_c\,. \tag{1.2}$$

(e) Prove the Jacobi identity by using the definition of the structure constants, working in the adjoint representation:

$${f^{ab}}_e{f^{ec}}_d + {f^{bc}}_e{f^{ea}}_d + {f^{ca}}_e{f^{eb}}_d = 0 \tag{1.3}$$

Operators that commute with every element of a Lie algebra are called *invariant Casimir operators*. It can be proved that there exist $(N-1)/2$ linearly-independent invariant Casimir operators for $\mathfrak{so}(N)$ if N is odd, $N/2$ if N is even, and $N-1$ for $\mathfrak{su}(N)$.

(f) For $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ compute the *quadratic Casimir operator*, defined as

$$C_2 = K_{ab}T^aT^b\,, \tag{1.4}$$

where K_{ab} is the inverse of K^{ab} (i.e. $K_{ab}K^{bc} = \delta_a^c$). Prove that it indeed commutes with every element of the corresponding Lie algebra.

[1] In this exercise a superscript can only be contracted with a subscript, therefore we will pay great attention to the position of the indices. Superscripts and subscripts can be treated as equal for all practical purposes.

Exercise 3.2: Covariant derivative **5P**

In general, the covariant derivative

$$D^\mu = \partial^\mu - \mathrm{i}gA^\mu = \partial^\mu - \mathrm{i}gA_a^\mu T^a \tag{2.1}$$

explicitely depends on the choice of the representation of the gauge group generators T^a. You know that in the fundamental representation

$$D'^\mu = UD^\mu U^{-1} \qquad \text{and} \qquad A'^\mu = UA^\mu U^{-1} - \frac{\mathrm{i}}{g}(\partial^\mu U)U^{-1}, \tag{2.2}$$

with $U = \exp(\mathrm{i}g\vartheta_a T_F^a)$.

(a) Prove that the form of the transformations above is vaild in any representation:

$$D'^\mu = VD^\mu V^{-1} \qquad \text{and} \qquad A'^\mu = VA^\mu V^{-1} - \frac{\mathrm{i}}{g}(\partial^\mu V)V^{-1}, \tag{2.3}$$

where $V = \exp(\mathrm{i}g\vartheta_a T^a)$, T being the generators in an arbitrary representation.

Hint: *Start from the transformation law for A^μ in the fundamental representation and extract the* exact *form of the gauge transformation for A_a^μ, independent of the representation of the generators. To this end, make use of the Baker–Campbell–Hausdorff formula:*

$$\exp(B)\,A\,\exp(-B) = \sum_{n=0}^{+\infty} \frac{1}{n!}A_n, \tag{2.4}$$

with $A_n = [B, A_{n-1}]$, $A_0 = A$, and remember that ${f^{ab}}_c = \mathrm{i}{(T_A^a)^b}_c$.

(b) Show that

$$F^{\mu\nu} = \frac{\mathrm{i}}{g}[D^\mu, D^\nu], \tag{2.5}$$

and is, therefore, representation-independent. Find how $F^{\mu\nu}$ transforms under a gauge transformation.

Exercise 3.3: Lorentz transformations **5P**

Taking $x \to x'^\mu = {\Lambda^\mu}_\nu x^\nu$ and the orthogonality condition, ${\Lambda^\mu}_\nu {\Lambda^\rho}_\sigma g_{\mu\rho} = g_{\nu\sigma}$,

(a) Show that $a^\mu b_\mu$ is invariant under Lorentz transformations.

Let ${(\Lambda^{-1})^\mu}_\nu$ be the inverse of ${\Lambda_\nu}^\mu$, meaning that

$${(\Lambda^{-1})^\mu}_\nu {\Lambda^\nu}_\rho = \delta_\rho^\mu = g_\rho^\mu\,. \tag{3.1}$$

(b) Show that ${(\Lambda^{-1})^\mu}_\nu = {\Lambda_\nu}^\mu$.

(c) Show that ${\partial'}_\mu = {(\Lambda^{-1})^\nu}_\mu \partial_\nu$ where ${\partial'}_\mu = \partial/\partial x'^\mu$.

(d) Does $M_{\nu\mu} = \pm M_{\mu\nu}$ imply ${M_\nu}^\mu = \pm {M_\mu}^\nu$, ${M_\nu}^\mu = \pm {M^\mu}_\nu$, or $M^{\nu\mu} = \pm M^{\mu\nu}$? If not, how should the metric tensor be modified to allow this?