Exercise 4.1: Valid Lagrangians **5P**

The typical recipe to construct a new model in particle physics is the following[1]:

1. Define a fundamental symmetry.
2. Define the particle and field content.
3. Construct a Lagrangian *density* $\mathcal{L}$ from all allowed combinations of particles and fields.

Any equations of motion can then be obtained via Hamilton's principle from the action

$$S = \int \mathrm{d}^4x\, \mathcal{L}. \tag{1.1}$$

Consider now the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + i\bar{\psi}(x)\gamma^\mu D_\mu(x)\psi(x) - m\bar{\psi}(x)\psi(x), \tag{1.2}$$

with the covariant derivative $D_\mu(x) = \partial_\mu + ieA_\mu(x)$ and the field-strength tensor $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. Make use of natural units throughout the exercise, *i.e.* when we say that something is *dimensionless* or *of dimension* q, we always refer to the dimension of energy.

(a) What are the requirements on $\mathcal{L}_{\text{QED}}$ to achieve a consistent description of QED?

(b) Determine the dimension of the action S, of the integration element d^4x and of the Lagrangian density $\mathcal{L}_{\text{QED}}$. Derive also the dimension of $\psi(x)$ and $F_{\mu\nu}(x)$.

Imagine now adding a scalar field $\varphi(x)$ to QED, which is a singlet under any gauge symmetry.

(c) Argue if the following terms would be allowed additions to $\mathcal{L}_{\text{QED}}$, and, if they are not, point out all the reasons why they are not allowed:

1. $\mathcal{L}_1 = g\varphi(x)\,\bar{\psi}(x)\,\psi(x)$;
2. $\mathcal{L}_2 = m\varphi(x)\,\overline{\psi}(x)\,\psi(x)$;
3. $\mathcal{L}_3 = i\varphi(x)\,A^\mu(x)\,A^\nu(x)$;
4. $\mathcal{L}_4 = \dfrac{1}{m} A_\mu(x)\,A^\mu(x)\,\overline{\psi}(x)\,\psi(x)$;
5. $\mathcal{L}_5 = \dfrac{g^2}{m}\partial_\mu A^\mu(x)\,\varphi(x)\,\dfrac{\partial\varphi(x)}{\partial t}$;
6. $\mathcal{L}_6 = \dfrac{1}{4} g^4 m^4$.

[1] Once the model is done you can add some phenomenological results as topping.

Exercise 4.2: Lorentz invariance of the Dirac Lagangian **8P**

To show the invariance of the Lagrangian density of a Dirac field,

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi, \tag{2.1}$$

we can use the chiral representation, where the Dirac spinor and the γ-matrices are given by

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}.$$

The Weyl spinors ψ_L and ψ_R transform under the $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ representation of the Lorentz group, respectively, and we have $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$, $\overline{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$.

(a) Show that the Lagrangian density can be decomposed into spinor products of the form

$$\psi_R^\dagger \sigma^\mu \psi_R, \qquad \psi_L^\dagger \overline{\sigma}^\mu \psi_L, \qquad \psi_R^\dagger \psi_L, \qquad \psi_L^\dagger \psi_R. \tag{2.2}$$

(b) Consider the Lorentz transforms of the spinors,

$$\psi_L \to \Lambda_L \psi_L = e^{(-i\vec{\vartheta} - \vec{\eta})\frac{\vec{\sigma}}{2}} \psi_L, \qquad \psi_R \to \Lambda_R \psi_R = e^{(-i\vec{\vartheta} + \vec{\eta})\frac{\vec{\sigma}}{2}} \psi_R,$$

to show that the terms in (a) either transform as a scalar or a vector

$$V^\mu \to \Lambda^\mu{}_\nu V^\nu = \begin{pmatrix} V_0 + \eta_i V_i \\ \vec{V} + \vec{\eta} V_0 + \vec{V} \times \vec{\vartheta} \end{pmatrix}. \tag{2.3}$$

(c) Use the results of the previous parts to show that the Lagrangian density of Eq. (2.1) is Lorentz invariant.

Exercise 4.3: Pauli-Lubanski pseudovector **7P**

The *Pauli-Lubanski pseudovector* describes the spin state of a moving particle:

$$W_\mu = \frac{1}{2} \tilde{M}_{\mu\sigma} P^\sigma = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma, \tag{3.1}$$

where $M^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)$ denotes the relativistic angular momentum tensor operator[2], and $P^\mu = i\partial^\mu$ is the 4-momentum. Its commutation relation is given as:

$$[W_\mu, W_\nu] = -i\varepsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma. \tag{3.2}$$

The simultaneous eigenvalues of P^2 and W^2 can be used to classify particles according to their mass and spin as irreducible representations of the Poincaré algebra.
We define the generalized Levi-Civita symbol in four dimensions as:

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 \text{ if } \{\mu,\nu,\rho,\sigma\} \text{ is an odd permutation of } \{0,1,2,3\} \\ -1 \text{ if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{0,1,2,3\} \\ 0 \text{ otherwise} \end{cases}, \tag{3.3}$$

with $\varepsilon^{0123} = g^{\mu 0} g^{\nu 1} g^{\rho 2} g^{\sigma 3} \varepsilon_{\mu\nu\rho\sigma} = -\varepsilon_{0123}$.

[2] This form of $M^{\mu\nu}$ is a generalization of the form for the generators of the Lorentz group given in the lecture, which is required once the operator acts on fields.

(a) Show that the components of W_μ for a particle at rest are $(0, -m\vec{J})^T$, where $\vec{J} = \vec{x} \times \vec{P}$ is the total angular momentum operator in three dimensions.

(b) Prove the following identities:

1. $[M_{\mu\nu}, P_\rho] = i\,(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu)$,
2. $W_\mu P^\mu = 0$,
3. $[W_\mu, P_\nu] = 0$.

(c) Prove that

$$[P^2, P_\mu] = 0\,, \qquad [P^2, M_{\mu\nu}] = 0\,, \qquad \text{and} \qquad [W^2, P_\mu] = 0\,. \tag{3.4}$$

These relations, together with $[W^2, M_{\mu\nu}] = 0$ show that P^2 and W^2 are the Casimir operators of the Poincaré group, since they commute with all its generators.