Exercise 5.1: Quantization of the complex scalar field 10P

The Hamiltonian of a complex-valued scalar field obeying the Klein–Gordon equation is given at the classical level by

$$H(t) = \int \mathrm{d}^3\vec{x} \left[\Pi^*(t,\vec{x})\Pi(t,\vec{x}) + \left(\vec{\nabla}\varphi^*(t,\vec{x})\right) \cdot \left(\vec{\nabla}\varphi(t,\vec{x})\right) + m^2\varphi^*(t,\vec{x})\varphi(t,\vec{x})\right]. \tag{1.1}$$

The field variables φ, φ^* can be written in terms of quantized normal modes as

$$\varphi(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} \left(a_{\vec{p}}\,\mathrm{e}^{-\mathrm{i}\,p\cdot x} + b^\dagger_{\vec{p}}\,\mathrm{e}^{+\mathrm{i}\,p\cdot x}\right), \tag{1.2}$$

$$\varphi^\dagger(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} \left(b_{\vec{p}}\,\mathrm{e}^{-\mathrm{i}\,p\cdot x} + a^\dagger_{\vec{p}}\,\mathrm{e}^{+\mathrm{i}\,p\cdot x}\right), \tag{1.3}$$

and are treated as independent quantities, with respective canonically conjugate momenta

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}, \tag{1.4}$$

$$\Pi^* = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi^*)}. \tag{1.5}$$

The theory is invariant under real phase transformations $\varphi' = \mathrm{e}^{\mathrm{i}\alpha}\varphi$, and $\varphi^{*\prime} = \mathrm{e}^{-\mathrm{i}\alpha}\varphi^*$, $\alpha \in \mathbb{R}$. This global U(1) symmetry leads to the Noether current j^μ at the classical level and the corresponding Noether charge:

$$Q(t) = \int \mathrm{d}^3\vec{x}\, j^0 = \mathrm{i}\int \mathrm{d}^3\vec{x}\,(\Pi^*(t,\vec{x})\varphi^*(t,\vec{x}) - \Pi(t,\vec{x})\varphi(t,\vec{x})). \tag{1.6}$$

In agreement with canonical quantization, the field operators are required to fulfill the commutation relations at equal time

$$\left[\hat{\varphi}(t,\vec{x}), \hat{\Pi}(t,\vec{y})\right] = \left[\hat{\varphi}^\dagger(t,\vec{x}), \hat{\Pi}^\dagger(t,\vec{y})\right] = \mathrm{i}\delta^{(3)}(\vec{x}-\vec{y}), \tag{1.7}$$

while the commutators involving the other combinations are required to vanish.

(a) Prove that the Heisenberg equations

$$\mathrm{i}\frac{\partial\hat{\varphi}(t,\vec{x})}{\partial t} = \left[\hat{\varphi}(t,\vec{x}), \hat{H}(t)\right], \tag{1.8}$$

$$\mathrm{i}\frac{\partial\hat{\Pi}(t,\vec{x})}{\partial t} = \left[\hat{\Pi}(t,\vec{x}), \hat{H}(t)\right], \tag{1.9}$$

imply that $\hat{\varphi}$ satisfies the Klein–Gordon equation.

(b) Using the field decomposition of Eq. (1.2) show that the classical Noether charge Q of Eq. (1.6) leads, at the quantum level, to the charge operator

$$\hat{Q}(t) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} \left(\hat{n}^{(a)}_{\vec{p}} - \hat{n}^{(b)}_{\vec{p}}\right), \tag{1.10}$$

with:

$$\hat{n}^{(a)}_{\vec{p}} = a^\dagger_{\vec{p}} a_{\vec{p}}\,, \qquad \text{and} \qquad \hat{n}^{(b)}_{\vec{p}} = b^\dagger_{\vec{p}} b_{\vec{p}} \tag{1.11}$$

and $E_p = \sqrt{\vec{p}^2 + m^2}$.

(c) Check that $\left[\hat{Q}(t), \hat{H}(t)\right] = 0$, namely that the charge is a conserved quantity in the quantized theory as well. To do this, recall that the expression of the quantized Hamiltonian (up to the zero-point energy term) is

$$\hat{H} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} E_p \left(\hat{n}^{(a)}_{\vec{p}} + \hat{n}^{(b)}_{\vec{p}}\right). \tag{1.12}$$

To investigate the action of $\hat{Q}$ on the field states, let us assume $|\alpha\rangle$ represents an eigenstate of $\hat{Q}$ with eigenvalue q, i.e. $\hat{Q}\,|\alpha\rangle = q\,|\alpha\rangle$.

(d) Show that

$$\hat{Q}\varphi^\dagger = \varphi^\dagger \left(\hat{Q} + 1\right) \tag{1.13}$$

$$\text{and thus} \quad \hat{Q}\varphi^\dagger\,|\alpha\rangle = (q+1)\varphi^\dagger\,|\alpha\rangle\,, \tag{1.14}$$

meaning that $\hat{\varphi}^\dagger$, can be viewed as an operator that increases the charge of a state by one unit.

Exercise 5.2: Properties of Dirac spinors **10P**

The momentum-space Dirac spinors $u_s(\vec{p})$ and $v_s(\vec{p})$ (with mass m, spin s, and momentum $\vec{p}$) are defined via the ansatz for plane wave solutions of the Dirac equation:

- $\psi_s(x) = u_s(\vec{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x}$ for positive energy solutions;
- $\psi_s(x) = v_s(\vec{p})\,\mathrm{e}^{\mathrm{i}p\cdot x}$ for negative energy solutions.

(a) Show that:

$$\not{p}u_s(\vec{p}) = +mu_s(\vec{p})\,, \qquad \bar{u}_s(\vec{p})\not{p} = +m\bar{u}_s(\vec{p})\,, \tag{2.1}$$

$$\not{p}v_s(\vec{p}) = -mv_s(\vec{p})\,, \qquad \bar{v}_s(\vec{p})\not{p} = -m\bar{v}_s(\vec{p})\,. \tag{2.2}$$

The $u_s(\vec{p})$ and $v_s(\vec{p})$ fulfill the orthogonality relations

$$\overline{u}_r(\vec{p})u_s(\vec{p}) = -\overline{v}_r(\vec{p})v_s(\vec{p}) = 2m\delta_{rs}\,, \tag{2.3}$$

$$\overline{u}_r(\vec{p})v_s(\vec{p}) = -\overline{v}_r(\vec{p})u_s(\vec{p}) = 0\,, \tag{2.4}$$

and the completeness relation

$$\sum_s \left[u_s(\vec{p})\overline{u}_s(\vec{p}) - v_s(\vec{p})\overline{v}_s(\vec{p})\right] = 2m. \tag{2.5}$$

(b) Check the completeness relation by showing that the basis states $u_r(\vec{p})$ and $v_r(\vec{p})$ are all eigenstates with the same eigenvalue.

Now we define the projection operators:

$$\Lambda^{\pm}(\vec{p}) = \frac{\pm\not{p} + m}{2m}, \tag{2.6}$$

which project the states of positive and negative energy, respectively, out of an arbitrary state:

$$f(\vec{p}) = \sum_s \left[\alpha_s u_s(\vec{p}) + \beta_s v_s(\vec{p})\right], \qquad \alpha, \beta \in \mathbb{C}\,. \tag{2.7}$$

(c) Show that $\Lambda^{\pm}(\vec{p})$ are indeed projectors, namely that

$$\left[\Lambda^{\pm}(\vec{p})\right]^2 = \Lambda^{\pm}(\vec{p})\,, \tag{2.8}$$

$$\Lambda^{+}(\vec{p}) f(\vec{p}) = \sum_s \alpha_s u_s(\vec{p}), \tag{2.9}$$

$$\Lambda^{-}(\vec{p}) f(\vec{p}) = \sum_s \beta_s v_s(\vec{p}), \tag{2.10}$$

$$\Lambda^{+}(\vec{p}) + \Lambda^{-}(\vec{p}) = 1. \tag{2.11}$$

(d) Using the previous results, show that

$$\sum_s u_s(\vec{p})\overline{u}_s(\vec{p}) = \not{p} + m\,, \tag{2.12}$$

$$\sum_s v_s(\vec{p})\overline{v}_s(\vec{p}) = \not{p} - m\,. \tag{2.13}$$

(e) Prove the following identity involving Dirac matrices

1. $\not{p}\not{q} = p \cdot q - \mathrm{i}\sigma^{\mu\nu} p_\mu q_\nu$;
2. $\gamma^\mu \gamma_\mu = 4$;
3. $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$;
4. $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\mu\nu}$;
5. $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu$;

where $\sigma^{\mu\nu} = (\mathrm{i}/2)\,[\gamma^\mu, \gamma^\nu]$.

(f) Prove the *Gordon identity*

$$\overline{u}_s(\vec{q})\gamma^\mu u_s(\vec{p}) = \overline{u}_s(\vec{q}) \left[\frac{q^\mu + p^\mu}{2m} + \frac{\mathrm{i}\sigma^{\mu\nu}(q-p)_\nu}{2m}\right] u_s(\vec{p})\,. \tag{2.14}$$