Exercise 6.1: Feynman propagator **2P**

The propagator of a field describes its probability amplitude for a propagation from one place to another and is an important ingredient in calculating scattering amplitudes. With the quantization of the fields as discussed in the lecture, the so-called Feynman propagator of a real scalar field $\varphi(x)$ is defined as the vacuum expectation value of the time-ordered product of the fields at different space-time points x and x',

$$i\Delta_F(x-x') \coloneqq \langle 0|T\varphi(x)\varphi(x')|0\rangle,$$

where $T\varphi(x)\varphi(x') \coloneqq \Theta(t-t')\varphi(x)\varphi(x') + \Theta(t'-t)\varphi(x')\varphi(x)$ defines the time-ordering operator. Upon several manipulations (which we do not consider here), the above expression takes the following form in momentum space:

$$\Delta_F(x) = \lim_{\epsilon\to 0^+} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon}. \tag{1.1}$$

Show that the Feynman propagator of Eq. (1.1) fulfills the inhomogeneous Klein-Gordon equation,

$$\left(\Box + m^2\right)\Delta_F(x) = -\delta^{(4)}(x),$$

where $\delta^{(4)}(x)$ denotes the 4-dimensional δ-distribution, being “defined” as

$$\delta^{(4)}(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip\cdot x}.$$

Exercise 6.2: Phase space **8P**

To calculate decay rates and cross sections we need an integration over the phase space of the particles in the final state. For a general process with two particles (momenta p_1, p_2, masses m_1, m_2) in the final state, this phase space integral is given by:

$$\int \mathrm{d}\Phi_2 = \int \frac{\mathrm{d}^3p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2),$$

where q is the four-momentum of the incoming particles. This integral then acts on the squared matrix element.

(a) Show that one can rewrite the integral as:

$$\int \mathrm{d}\Phi_2 = \int \mathrm{d}\Omega \frac{1}{32\pi^2 q^2} \lambda(q^2, m_1^2, m_2^2)\Theta(q_0)\Theta(q^2 - (m_1+m_2)^2)$$

where we have used the Källén function:

$$\lambda(a^2, b^2, c^2) = \sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2} = \sqrt{(a^2 - b^2 - c^2)^2 - 4b^2c^2},$$

and the Heavyside step function Θ, and $\mathrm{d}\Omega = \mathrm{d}(\cos\vartheta_1)\mathrm{d}\varphi_1$ is the integration over the solid angle of particle 1 in the centre-of-mass frame of the two-particle system. The function λ describes the momentum of both particles in the centre-of-mass frame:

$$|\vec{p}_1|^2 = |\vec{p}_2|^2 = \frac{\lambda(q^2, m_1^2, m_2^2)}{2\sqrt{q^2}}$$

Hints:

- Use the relation:

$$\frac{\mathrm{d}^3 p}{2E} = \mathrm{d}^4 p \Theta(p_0)\delta(p^2 - m^2)$$

- Work in the centre-of-mass frame of the two final-state particles. Justify this!

(b) Give the explicit result for $\mathrm{d}\Phi_2$ in the cases $m_1 = m_2 = m$, $m_1 = m_2 = 0$ and $m_1 = 0 \neq m_2$.

Exercise 6.3: QED scattering $\boxed{\textbf{10P}}$

In this exercise, you will discuss two important processes in QED: eletron-positron annihilation into photons, which was discussed in the lecture, as well as electron-photon scattering, also known as Compton scattering.

(a) Consider first the process of electron-positron annihilation into two photons,

$$e^-(p_1) + e^+(p_2) \to \gamma(k_1) + \gamma(k_2).$$

Draw all Feynman diagrams contributing at leading order. Label them with the momenta of the internal propagators and the wave function factors for the external particles (including their momenta).

(b) Use the Feynman rules of QED to write down the contribution of the diagrams of (a) to the scattering amplitude. You should obtain the following result:

$$\begin{aligned} i\mathcal{M}_{e^-e^+} &= i\mathcal{M}_{e^-e^+,t} + i\mathcal{M}_{e^-e^+,u} \\ &= ie^2 \varepsilon_\mu^{(\lambda_1)*}(k_1)\, \varepsilon_\nu^{(\lambda_2)*}(k_2)\, \bar{v}_{s_2}(p_2) \left[\frac{2p_1^\mu\gamma^\nu - \gamma^\nu \not{k}_1 \gamma^\mu}{2p_1 \cdot k_1} + \frac{2p_1^\nu\gamma^\mu - \gamma^\mu \not{k}_2 \gamma^\nu}{2p_1 \cdot k_2} \right] u_{s_1}(p_1), \end{aligned}$$

where s_1, s_2 (λ_1, λ_2) are the spins (polarizations) of the electrons (photons).
Note: The on-shell relations $p_1^2 = p_2^2 = m^2$ (with m the mass of the electron and positron) and $k_1^2 = k_2^2 = 0$, the anticommutation relation for γ matrices as well as the Dirac equations for the spinors might be helpful for simplifying the amplitude.

(c) A correction to the process of (a) is given by an additional emission of a photon into the final state, leading to the process

$$e^-e^+ \to \gamma\gamma\gamma.$$

Draw the corresponding Feynman diagrams for this process (you do not need to add labels for the momenta). Can you guess how many diagrams there are for n photons in the final state?

(d) A related process is Compton scattering, which is the dominant interaction between electromagnetic radiation and matter for photon energies of roughly 100 keV to 10 MeV. It is described by the inelastic scattering process:

$$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k').$$

Draw the leading-order Feynman diagrams for this process. How do they differ to the ones from part (a)?

(e) Write down the scattering amplitude for the diagrams of (d). You should obtain:

$$\begin{aligned} i\mathcal{M}_{e^-\gamma} &= i\mathcal{M}_{e^-\gamma,s} + i\mathcal{M}_{e^-\gamma,u} \\ &= -ie^2 \varepsilon_\mu^{(\lambda)}(k)\, \varepsilon_\nu^{(\lambda')*}(k')\, \bar{u}_{s'}(p') \left[\frac{2p^\mu\gamma^\nu + \gamma^\nu \not{k} \gamma^\mu}{2p\cdot k} - \frac{2p^\nu\gamma^\mu - \gamma^\mu \not{k}' \gamma^\nu}{2p\cdot k'} \right] u_s(p), \end{aligned}$$

where again s, s' (λ, λ') are the spins (polarizations) of the electrons (photons). Can you think of a set of rules (for the momenta, spinors, ...) to relate this scattering amplitude to the result of part (b)?

(f) Is the scattering of four photos, i.e. $\gamma\gamma \to \gamma\gamma$, possible withing QED?