Due to a lack of participants in the Monday tutorial sessions of the last weeks, we will only keep the Tuesday, 9:45 h session further on.

Exercise 7.1: Gammanastics 10P

As you saw in the lectures, the calculation of scattering amplitudes quite often introduces traces in the matrix element expressions. Since all fermions are spinor objects, these traces encompass chains of γ-matrices. Luckily, thanks to the Clifford-algebra obeyed by them, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbb{1}_4$, a number of helpful identities can be proven by you in the following. Do not choose a specific representation.

(a) Under hermitian conjugation the γ-matrices behave as $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$. Show that in general this can be written as $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

(b) It has been found convenient to also introduce a fifth matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \,.$$

Show the equivalence of the following expression:

$$\gamma^5 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \,.$$

(c) Show the following identities:
  (i) $\left(\gamma^5\right)^\dagger = \gamma^5$,
  (ii) $\left(\gamma^5\right)^2 = \mathbb{1}_4$,
  (iii) $\{\gamma^\mu, \gamma^5\} = 0$.

(d) Calculate these traces:
  (i) $\mathrm{Tr}[\gamma^\mu] = 0$,
  (ii) $\mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] = 0$, for any odd n,
  (iii) $\mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] = \mathrm{Tr}[\gamma^{\mu_n} \cdots \gamma^{\mu_1}]$,
  (iv) $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$,
  (v) $\mathrm{Tr}[\gamma^5] = 0$,
  (vi) $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^5] = 0$,
  (vii) $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] = -4i\epsilon^{\mu\nu\rho\sigma}$.

Exercise 7.2: Bhabha Scattering Amplitude 10P

Consider the $2 \to 2$ scattering process

$$e^+e^- \to e^+e^-, \tag{2.1}$$

where the electrons and positrons may be considered massless.

(a) Draw all Feynman diagrams contributing to the process at tree level in QED and write down the associated matrix elements.

(b) Write down the expression for the spin-averaged squared matrix element.
Hint: think carefully about the relative sign between the matrix elements, in their sum.

(c) Perform the spin averaging. You can make use of your results from the previous exercise and the identity

$$\sum_{s_i,s_j}(\bar{u}(p_i,s_i)\Gamma_1 u(p_j,s_j))(\bar{u}(p_i,s_i)\Gamma_2 u(p_j,s_j))^\dagger = \mathrm{Tr}\left[\Gamma_1(\not{p}_j + m_j)\bar{\Gamma}_2(\not{p}_i + m_i)\right] , \quad (2.2)$$

where the $\Gamma_{1,2}$ is an arbitrary chain of Gamma matrices and $\bar{\Gamma} = \gamma_0\Gamma^\dagger\gamma_0$. (What would need to change to accomodate spinor chains with $v(p_i, s_i)$?). Express your result in the Mandelstam variables

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^2 \qquad u = (p_1 - p_4)^2 \ .$$

Note the dependence of these definitions on your momenta labeling.

(d) Working in the centre-of-mass frame, rewrite your result in terms of the centre-of-mass energy $\sqrt{s}$ and the scattering angle between the incoming and outgoing particles, ϑ.

(e) The differential cross section has a form like

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \sum_{i,j\neq i} \mathcal{M}_i^\dagger\mathcal{M}_i + \mathcal{M}_i^\dagger\mathcal{M}_j \ , \quad (2.3)$$

where the first part describes the contribution of all individual Feynman diagrams $\mathcal{M}_i$ and the second part the interference of various diagrams. Plot $\mathrm{d}\sigma/\mathrm{d}\Omega$ as a function of ϑ for a value $\sqrt{s} = 10.58$GeV, including also curves showing separately the contribution from each Feynman diagram and their interference.

(f) Plot $\mathrm{d}\sigma/\mathrm{d}\Omega$ for at least 3 different values of $s \in [0, 300]$ GeV. What do you observe?