Introduction to Theoretical Particle Physics
Exercise Sheet 8
Winter term 2023/24
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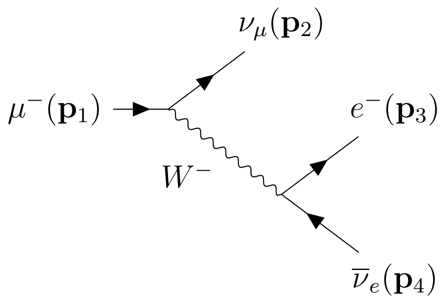
Exercise 8.1: Muon decay **20P**

In the Standard Model, the decay of a muon into an electron and (anti)neutrinos

$$\mu^-(\mathbf{p}_1) \to \nu_\mu(\mathbf{p}_2) + e^-(\mathbf{p}_3) + \overline{\nu}_e(\mathbf{p}_4)\,, \tag{1.1}$$

is mediated by a W boson, as shown by the tree level diagram



The matrix element for this process is given by[1]

$$\mathrm{i}\,\mathcal{M}_{s_1s_2s_3s_4} = \overline{u}_{s_2}(\mathbf{p}_2)\frac{\mathrm{i}g}{\sqrt{2}}\gamma^\mu\frac{1-\gamma^5}{2}u_{s_1}(\mathbf{p}_1)\ \frac{-\mathrm{i}g_{\mu\nu}}{q^2-M_W^2}\ \overline{u}_{s_3}(\mathbf{p}_3)\frac{\mathrm{i}g}{\sqrt{2}}\gamma^\nu\frac{1-\gamma^5}{2}v_{s_4}(\mathbf{p}_4)\,, \tag{1.2}$$

where g (with no Lorentz indices) is the weak coupling and M_W is the W boson mass. The decay happens at energy scales comparable to the muon mass m, so we can assume

$$m^2 \sim q^2 \ll M_W^2 \tag{1.3}$$

and neglect the masses of the electron and of the neutrinos. Expanding the W propagator in $q^2 \sim 0$,

$$\frac{1}{q^2-M_W^2} = -\frac{1}{M_W^2}\frac{1}{1-\frac{q^2}{M_W^2}} = -\frac{1}{M_W^2}\left[1+\mathcal{O}\left(q^2/M_W^2\right)\right], \tag{1.4}$$

the amplitude reduces to an effective four fermion vertex. This leads to

$$\mathcal{M}_{s_1s_2s_3s_4} = -\frac{G_F}{\sqrt{2}}\ \overline{u}_{s_2}(\mathbf{p}_2)\gamma^\mu(1-\gamma^5)u_{s_1}(\mathbf{p}_1)\ \overline{u}_{s_3}(\mathbf{p}_3)\gamma_\mu(1-\gamma^5)v_{s_4}(\mathbf{p}_4)\,, \tag{1.5}$$

where we introduced the *Fermi constant* $G_F/\sqrt{2} = g^2/(8M_W^2)$.

[1] The factors $\frac{1-\gamma^5}{2}$ in the matrix element project out the left-handed spin-configurations of the fermions.

(a) Show that, after averaging over incoming and summing over outgoing states, the modulus square of the matrix element can be expressed in terms of traces over Dirac matrices as

$$\overline{|\mathcal{M}|^2} = \frac{G_F^2}{4}\ \mathrm{Tr}\left[\not{p}_2\gamma^\mu(1-\gamma^5)(\not{p}_1+m)\gamma^\nu(1-\gamma^5)\right]\ \mathrm{Tr}\left[\not{p}_3\gamma_\mu(1-\gamma^5)\not{p}_4\gamma_\nu(1-\gamma^5)\right]. \tag{1.6}$$

Hint: *Remember that*

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \not{p} + m\,, \qquad \sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \not{p} - m\,. \tag{1.7}$$

(b) Calculate the traces and show that

$$\overline{|\mathcal{M}|^2} = 64\, G_F^2\,(p_1\cdot p_4)\,(p_2\cdot p_3)\,. \tag{1.8}$$

Hint: *Additional identities that you might find useful are*

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] = 4\mathrm{i}\epsilon^{\mu\nu\rho\sigma} \qquad \textit{and} \qquad \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2\left(g^\mu_\rho g^\nu_\sigma - g^\mu_\sigma g^\nu_\rho\right). \tag{1.9}$$

We now use this result to calculate the decay width, whose differential form is given by

$$\mathrm{d}\Gamma = \frac{1}{2E_1}\overline{|\mathcal{M}|^2}\,\mathrm{d}\Phi_3\,, \tag{1.10}$$

with the three particle phase space

$$\mathrm{d}\Phi_3 = \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\mathbf{p}_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^{(4)}(p_1-p_2-p_3-p_4)\,. \tag{1.11}$$

It is advantageous to work in the rest frame of the muon, i.e. $p_1 = (m, \mathbf{0})$.

(c) Express $\overline{|\mathcal{M}|^2}$ in terms of the muon mass m and of the energy E_4 of the antineutrino.

(d) Use the Dirac delta function to perform the $\mathrm{d}^3\mathbf{p}_2$ integration in the phase-space $\mathrm{d}\Phi_3$. Momentum conservation is implicit from this point on, so you are not allowed to use equalities containing p_2 any longer. Carry out all the remaining integrations except for those over the energies E_3 and E_4.

(e) Explicitly fix the boundaries for the E_3 and E_4 integrals using the Dirac delta and Heaviside theta functions.

Hint: *In the extremal cases, where one of the three final state momenta goes to zero, the kinematics of the remaining particles corresponds to a two-body decay.*

(f) Calculate the energy spectrum $\mathrm{d}\Gamma/\mathrm{d}E_3$ of the emitted electron.

(g) Calculate the total decay width Γ.
Compare the lifetime $\tau = \hbar/\Gamma$ of the muon with its experimental value.

The relevant numerical values are (R.L. Workman *et al.*, *Review of Particle Physics*, 2022):

$$\begin{aligned} \tau &= 2.196\,981\,1(22)\times 10^{-6}\,\mathrm{s}\,,\\ m &= 0.105\,658\,375\,5(23)\,\mathrm{GeV}\,,\\ M &= 80.377(12)\,\mathrm{GeV}\,,\\ G_F &= 1.166\,378\,7(6)\times 10^{-5}\,\mathrm{GeV}^{-2}\,,\\ \hbar &= 6.582\,119\,569\times 10^{-25}\,\mathrm{GeV\,s}\,. \end{aligned}$$