Introduction to Theoretical Particle Physics
Exercise Sheet 9
Winter term 2023/24
Lecturers: Prof. Dr. G. Heinrich, Dr. M. Kerner
Tutors: Dr. M. Bonetti, A. Vestner

Issued: 22.12.2023
Due: 12.01.2024

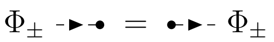
- There will be **NO** exercise class on Tuesday 09.01.2024.
- Solutions for Exercise Sheets 8 and 9 will be uploaded on ILIAS on 12.01.2024 and a Q&A session on both sheets will take place on Tuesday 16.01.2024.
- We kindly ask you to submit solutions for Exercise Sheet 8 and 9 **online** on ILIAS.

Exercise 9.1: Human interactions during Christmas time **10 P + 5 P**

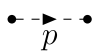
We can model human beings in terms of a complex scalar field of mass m, described by the Lagrangian density

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \,, \tag{1.1}$$

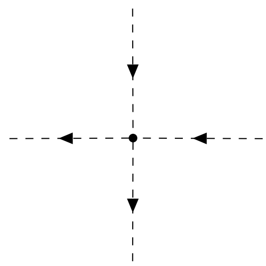
giving raise to the well-know Feynman rules



$= 1\,,$



$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}0^+}\,,$


$= \mathrm{i}\,\lambda\,,$

(a) Calculate the differential cross section $\mathrm{d}\sigma/\mathrm{d}\cos\vartheta$ for human-human interaction at LO (i.e. tree level)

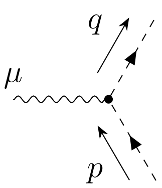
$$\Phi(\mathbf{p}_1)\Phi(\mathbf{p}_2) \to \Phi(\mathbf{p}_3)\Phi(\mathbf{p}_4) \tag{1.2}$$

averaging over initial states and summing over final states.

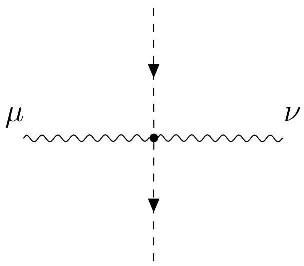
Strong experimental evidences show that Christmas has a strong effect on human-human interactions. We can use a U(1) gauge field A to model Christmas (notice how A is shaped like a Christmas tree), and couple it to the human field through the covariant derivative as

$$\partial_\mu \quad \to \quad D_\mu = \partial_\mu - \mathrm{i} g A_\mu \,. \tag{1.3}$$

This coupling provides the additional Feynman rules



$= \; -\mathrm{i}\, g\, (p^\mu + q^\mu)\,,$

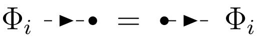


$= 2\mathrm{i}\, g^2 g^{\mu\nu}\,,$

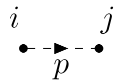
plus the usual Feynman rules for a U(1) gauge field.

(b) Calculate again $\mathrm{d}\sigma/\mathrm{d}\cos\vartheta$ considering the additional interactions coming from the U(1) Christmas field.[1] Sketch its angular behavior in the $\lambda \gg g$ limit (strong bonds) and in the $g \gg \lambda$ limit (formal bonds).

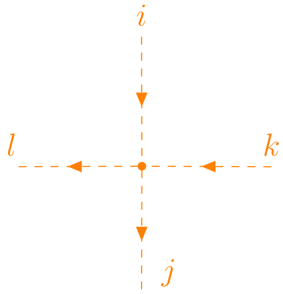
[Facultative: +5 points] A colorful Christmas is a merrier Christmas: if we replace the U(1) gauge symmetry with the QCD SU(3) gauge symmetry we get a Φ_i, $i = 1, 2, 3$ complex scalar triplet (i is the color index in the fundamental representation) and the new Feynman rules



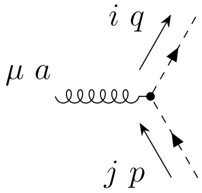
$= 1\,,$



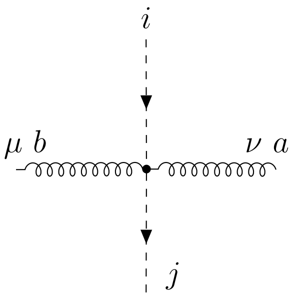
$$= \frac{\mathrm{i}\,\delta^{ij}}{p^2 - m^2 + \mathrm{i}0^+}\,,$$



$$= \mathrm{i}\,\lambda(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})\,,$$



$$= -\mathrm{i}\,g\,(p^\mu + q^\mu)\,(T^a)_{ij}\,,$$



$$= 2\mathrm{i}\,g^2 g^{\mu\nu} (T^a)_i{}^k (T^b)_{kj}$$

together with the usual Feynman rules of QCD.

(c) Calculate again $\mathrm{d}\sigma/\mathrm{d}\cos\vartheta$ using the colorful Christmas model.[2] Sketch its angular behavior in the $\lambda \gg g$ limit (strong bonds) and in the $g \gg \lambda$ limit (formal bonds).

Exercise 9.2: Proton form factor **10P**

The proton, a spin 1/2 particle, can be represented as a Dirac field of mass m. It is a bound state with a complex inner structure, therefore the matrix element describing its coupling to the electromagnetic field is more involved than the one derived for electrons.
The most general expression of a parity conserving matrix element between the proton and the electromagnetic current is

$$\langle \mathbf{p}', \lambda' | j^\mu(x) | \mathbf{p}, \lambda \rangle = e\, \mathrm{e}^{-\mathrm{i}(p-p')\cdot x}\, \overline{u}_{\lambda'}(\mathbf{p}') \left[p^\mu A + p'^\mu B + \gamma^\mu C + \sigma^{\mu\nu} p_\nu D + \sigma^{\mu\nu} p'_\nu E \right] u_\lambda(\mathbf{p})\,. \quad (2.1)$$

where $A, \ldots, E$ are Lorentz invariant complex functions of q^2 and m (often called *form factors*) and $q = p' - p$ is the momentum transferred to the proton during the interaction.

(a) Prove that

$$\overline{u}(\mathbf{p}')\,[(p' - p)^\mu + \mathrm{i}\sigma^{\mu\nu}(p' + p)_\nu]\,u(\mathbf{p}) = 0 \quad (2.2)$$

[1]This theory is called *scalar Quantum Electrodynamics* (sQED).
[2]This model is called *scalar Quantum Chromodynamics* (sQCD).

and use this identity together with the Gordon identity derived in `Exercise 5.2` to show that A and B can be absorbed into C, D and E, such that without loss of generality we can set $A = B = 0$.

(b) The current is hermitian, therefore $j^{\dagger\mu}(x) = j^\mu(x)$. Use this property to deduce relations between the real and imaginary parts of C, D and E.

(c) Which further constraints follow from the conservation law $\partial_\mu \langle \mathbf{p}', \lambda' | j^\mu(x) | \mathbf{p}, \lambda \rangle = 0$?

(d) Making use of the information acquired in previous points, show that Eq. (2.1) can be written as

$$\langle \mathbf{p}', \lambda' | j^\mu(x) | \mathbf{p}, \lambda \rangle = e\, \mathrm{e}^{-\mathrm{i}(p-p')\cdot x}\, \overline{u}_{\lambda'}(\mathbf{p}') \left[\gamma^\mu F_1\left(\frac{q^2}{m^2}\right) + \mathrm{i}\frac{\sigma^{\mu\nu} q_\nu}{2m} F_2\left(\frac{q^2}{m^2}\right) \right] u_\lambda(\mathbf{p})\,, \quad (2.3)$$

with two real functions F_1 and F_2.
Give the explicit relations between C, D, E and F_1, F_2.