Exercise B.1: Unitarity in $\nu\nu \to WW$ scattering **15 P**

Consider the production of longitudinally polarized W bosons from neutrino-antineutrino annihilation

$$\nu_e(\vec{p}_1)\,\overline{\nu}_e(\vec{p}_2) \;\to\; W_0^+(\vec{p}_3)\,W_0^-(\vec{p}_4)\,, \tag{1.1}$$

where the subscript 0 indicates that we are considering longitudinally polarized W bosons.

(a) Draw the (two) Feynman diagrams that describe this process at tree-level in the Standard Model. For a complete list of Feynman rules in the Standard Model, use `arxiv.org/abs/1209.6213`.[1]

(b) Write down the scattering amplitude. Check that it is given by the sum of the following two terms:

$$\mathcal{M}_s = \frac{-e^2}{4\sin^2\vartheta_W}\frac{1}{s-m_Z^2}\left[\overline{v}(\vec{p}_2)\gamma_\alpha(1-\gamma_5)u(\vec{p}_1)\right]\Gamma^{\mu\nu\alpha}\epsilon^{*0}_{\mu}(\vec{p}_3)\epsilon^{*0}_{\nu}(\vec{p}_4)\,, \tag{1.2}$$

$$\mathcal{M}_t = \frac{-e^2}{4\sin^2\vartheta_W}\frac{1}{t}\left[\overline{v}(\vec{p}_2)\gamma^\nu(\not{p}_1-\not{p}_3)\gamma^\mu(1-\gamma_5)u(\vec{p}_1)\right]\epsilon^{*0}_{\mu}(\vec{p}_3)\epsilon^{*0}_{\nu}(\vec{p}_4)\,, \tag{1.3}$$

where ϑ_W is the Weinberg mixing angle, $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, and all the leptons are massless. Calculate the explicit expression $\Gamma^{\mu\nu\alpha}$.

The longitudinal polarization of an on-shell gauge boson with mass m is defined as

$$\epsilon^{0,\mu}(\vec{k}) = \gamma\begin{pmatrix}|\vec{\beta}|\\ \hat{\beta}\end{pmatrix}\,, \tag{1.4}$$

with

$$\vec{\beta}=\frac{\vec{k}}{k^0}\,,\qquad \gamma=\frac{1}{\sqrt{1-\vec{\beta}^2}}\,,\qquad \hat{\beta}=\frac{\vec{\beta}}{|\vec{\beta}|} \tag{1.5}$$

(c) Show that in the center of mass frame the longitudinal polarizations of the W bosons can be written as

$$\epsilon^{*0,\mu}(\vec{p}_3)=\frac{\sqrt{s}}{2m_W}\begin{pmatrix}\sqrt{1-\frac{4m_W^2}{s}}\\ \sin\vartheta\\ 0\\ \cos\vartheta\end{pmatrix}\,,\qquad \epsilon^{*0,\mu}(\vec{p}_4)=\frac{\sqrt{s}}{2m_W}\begin{pmatrix}\sqrt{1-\frac{4m_W^2}{s}}\\ -\sin\vartheta\\ 0\\ -\cos\vartheta\end{pmatrix}\,. \tag{1.6}$$

Here ϑ is defined as the scattering angle between the incoming neutrino ν_e and the outgoing W^+ in the center of mass frame.

[1] There are codes that determine the Feynman diagrams contributing to n amplitude, e.g. `qgraf`.

An explicit representation of a Dirac fermion is given by

$$u^+(\vec{p}) = \frac{\not{p}+m}{\sqrt{p^0+m}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad u^-(\vec{p}) = \frac{\not{p}+m}{\sqrt{p^0+m}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \tag{1.7}$$

$$v^+(\vec{p}) = \frac{\not{p}-m}{\sqrt{p^0+m}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \qquad v^-(\vec{p}) = \frac{\not{p}-m}{\sqrt{p^0+m}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \tag{1.8}$$

with the gamma matrices in the Dirac representation

$$\gamma^0 = \begin{pmatrix}\mathbb{1}_{2\times2} & 0\\ 0 & -\mathbb{1}_{2\times2}\end{pmatrix}, \qquad \gamma^k = \begin{pmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix}0 & \mathbb{1}_{2\times2}\\ \mathbb{1}_{2\times2} & 0\end{pmatrix}, \tag{1.9}$$

and the Pauli matrices σ^k.

(d) Use this representation to check that the two scattering amplitudes $\mathcal{M}_{s,t}$ behave like $\mathcal{M}_{s,t} \sim s$ in the limit $s \to \infty$.

Hint: *Show that* $\epsilon^{*0,\mu}(\vec{p}_{3,4}) \sim p^\mu_{3,4}/m_W$ *for* $s \to \infty$.

(e) How does the total amplitude $\mathcal{M}_s + \mathcal{M}_t$ behave in the limit of high energies? Interpret your result.

Exercise B.2: Massive QED & Stückelberg Mechanism **5 P**

Consider QED with only one fermion ψ and the photon field A according to the Lagrangian

$$\mathcal{L}_{\text{QED}} = \overline{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{2.1}$$

with

$$D_\mu = \partial_\mu + \mathrm{i}eA_\mu\,, \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,, \tag{2.2}$$

where the fields transform as

$$\psi(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x)\,, \qquad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)\,. \tag{2.3}$$

(a) Explicitly show that $\mathcal{L}_{\text{QED}}$ is gauge invariant, i.e. does not change if the above transformation is applied. On the other hand, show that the term

$$\mathcal{L}_{\text{mass}} = \frac{m^2}{2}A^\mu A_\mu \tag{2.4}$$

is not invariant.

Gauge invariance can be achieved by adding to $\mathcal{L}_{\text{QED}}$ a term of the form

$$\mathcal{L}_{\text{S}} = \frac{m_S^2}{2}\left(A^\mu + \frac{1}{m_S}\partial^\mu\sigma\right)\left(A_\mu + \frac{1}{m_S}\partial_\mu\sigma\right), \tag{2.5}$$

with an additional scalar field σ.

(b) How must $\sigma(x)$ transform so that $\mathcal{L}_\mathrm{S}$ is gauge invariant?

The additional term

$$\mathcal{L}_\mathrm{G} = -\frac{1}{2\xi}(\partial^\mu A_\mu + m_S \xi \sigma)^2 \,, \tag{2.6}$$

with $\xi > 0$, is required for a complete theory.

(c) What constraint should α satisfy to have a gauge invariant $\mathcal{L}_\mathrm{G}$?