Introduction to Theoretical Particle Physics
Exercise Sheet C
Winter term 2023/24
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Exercise C.1: The dark side of the photons [10 P]

Here we consider a model which is beyond the Standard Model. We add three terms to the Standard Model (SM) Lagrange density,

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}} - \frac{1}{4} F^V_{\mu\nu} F^{V\mu\nu} + \frac{1}{2} m_V^2 V_\mu V^\mu + \frac{\epsilon}{2} F^V_{\mu\nu} F_B^{\mu\nu} \,, \tag{1.1}$$

where $F^V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. The first additional term is the kinetic term of a new $U(1)$ gauge boson V_μ (let's call this a "dark photon"). The second term is the mass term of the dark photon, and the third term is a kinetic mixing of the dark photon and the hypercharge $U(1)$ gauge boson B_μ of the SM. If $\epsilon = 0$, the dark photon decouples from the SM completely. Here we assume $0 < \epsilon \ll 1$ and $m_V \ll vg$.

(a) By making a replacement $B_\mu \to B_\mu + \epsilon V_\mu$ and neglecting $\mathcal{O}(\epsilon^2)$ terms, we can diagonalize the kinetic terms of B_μ and V_μ. Then the mass terms of W^3_μ, B_μ and V_μ are expressed as a matrix,

$$\text{Eq. (1.1)} \ni \frac{1}{2} m^2 \begin{pmatrix} W^3_\mu & B_\mu & V_\mu \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} W^{3\mu} \\ B^\mu \\ V^\mu \end{pmatrix} , \tag{1.2}$$

where $m = \frac{1}{2} v \sqrt{g^2 + g'^2}$. Write down the mass matrix including terms of $\mathcal{O}(\epsilon)$.

(b) Diagonalize the mass matrix obtained in the previous question, neglecting $\mathcal{O}(\epsilon^3)$, $\mathcal{O}(m_V^3)$, and $\mathcal{O}(\epsilon^2 m_V^2)$ terms.

(c) The interactions of the electron and gauge bosons are given by the terms

$$\text{Eq. (1.1)} \ni \bar{e}_L \left[-\frac{1}{2}(g W^3_\mu + g' B_\mu)\gamma^\mu \right] e_L - g' \bar{e}_R B_\mu \gamma^\mu e_R \,, \tag{1.3}$$

where e_L is the left-handed electron field and e_R is the right-handed electron field. The usual Dirac field of the electron is given by $\psi \equiv \begin{pmatrix} e_R \\ e_L \end{pmatrix}$.
By making the replacement $B_\mu \to B_\mu + \epsilon V_\mu$ and the subsequent diagonalization of the mass matrix, an interaction between the electron and the dark photon is induced. Write down this interaction term, neglecting $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(m_V)$ terms.

Exercise C.2: Time to hit the GIM [5 P]

Imagine you were a particle physicist at the end of the 1960s: only the three light quarks u, d, and s are known to exist. Their electro-weak interaction is described by

$$\mathcal{L} = -\frac{g}{2\sqrt{2}} W_\mu^+ J^\mu_{\mathrm{CC}} - \frac{g}{2\cos\vartheta_W} Z_\mu J^\mu_{\mathrm{NC}}$$
$$J^\mu_{\mathrm{CC}} = \overline{u}\,\gamma^\mu\,(1-\gamma_5)\,d'\ ; \qquad J^\mu_{\mathrm{NC}} = \overline{u}\,\gamma^\mu\,(g_v - g_a\,\gamma_5)\,u + \overline{d}'\,\gamma^\mu\,(g_v - g_a\,\gamma_5)\,d'\ , \tag{2.1}$$

with the weak eigenstate $d' = d\cos(\vartheta_C) + s\sin(\vartheta_C)$ and the mass eigenstates d and s of down- and strange-quark, respectively. ϑ_C is the Cabbibo-angle.

(a) Show that the charged currents J^μ_{CC} as well as the neutral currents J^μ_{NC} would lead to flavour-changing processes, e.g. interactions leading to $\Delta S = +1$, and draw the corresponding Feynman diagram.

To your utter bewilderment, experimental data lack any hint on the existence of these flavour-changing neutral currents (FCNCs). After puzzeling over this for some years you are delighted to find a neat article in your bed-time reading by S.L. Glashow, J. Iliopoulos, and L. Maiani, who postulate the existence of a fourth quark they call the “charm quark”.

(b) As a matter of fact, you are so excited about it that you cannot help yourself, but to investigate the thoroughness of this new model. Check whether direct FCNC couplings are existent after the introduction of a second quark doublet

$$\begin{pmatrix} u \\ d' \end{pmatrix} = \begin{pmatrix} u \\ d\cos(\vartheta_C) + s\sin(\vartheta_C) \end{pmatrix} \qquad \begin{pmatrix} c \\ s' \end{pmatrix} = \begin{pmatrix} c \\ dX + sY \end{pmatrix} .$$

Determine the suitable values for X and Y and show the unitarity of the mixing matrix

$$V_{dd',ss'} = \begin{pmatrix} \cos(\vartheta_C) & \sin(\vartheta_C) \\ X & Y \end{pmatrix} . \tag{2.2}$$

Hint: remember to adjust J_{NC} for the additional doublet.

Exercise C.3: The Unitarity Triangle **5 P**

Just as the Bermuda triangle seems to have swallowed an inappropriate amount of ships and planes, the Unitarity Triangle (UT) seems to have swallowed an inappropriate amount of hope for new physics (at least from flavour physicists). To also give you the opportunity to despair[1], some exercises to get acquainted with the CKM-matrix and the UT are just the right thing.
As seen in the lecture, the CKM-matrix V_{CKM} is the basis-change operator for the down-type quarks between the mass eigenbasis and the flavour (or weak) eigenbasis, i.e.

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V_{\mathrm{CKM}} \cdot \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \cdot \begin{pmatrix} d \\ s \\ b \end{pmatrix} . \tag{3.1}$$

The 9 complex elements of the matrix itself cannot be predicted by theory and must thus be determined experimentally. However, it is possible to show that only four parameters are required to parametrize the matrix, as, e.g., the PDG parameterization does:

$$V_{\mathrm{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} \tag{3.2}$$

[1] of the absence of clear hints towards new physics in the flavour sector

This parameterization contains three mixing angles $\{\vartheta_{12}, \vartheta_{23}, \vartheta_{13}\}$ for the three generations and a CP violating complex phase δ. The occurring s_{ij} and c_{ij} are $\sin(\vartheta_{ij})$ and $\cos(\vartheta_{ij})$ respectively.

(a) Experimental results show that V_{CKM} is almost the identity matrix. Introduce the *Wolfenstein paramterization*

$$s_{12} = \lambda \qquad s_{23} = A\lambda^2 \qquad s_{13}e^{\mathrm{i}\delta} = A\lambda^3(\rho + \mathrm{i}\eta)$$

and expand each element of the matrix in λ up to order three.
Hint: You can assume $\delta < \pi/2$ and $\lambda \ll 1$, i.e. $a\lambda + b\lambda^3 \approx a\lambda$.

(b) To see the origin of the term *triangle*, use the unitarity condition on the CKM-matrix in eq. (3.1). Which of the equations describe a triangle in the complex plane?

(c) Use the Wolfenstein parameterization to sketch the triangles. Which of them do you recon to be experimentally the most accessible one?

The UT has two important implications:

- The SM contains CP violation (CPV), if the triangle is not degenerated.
- There are only 3 generations of quarks, if the triangle is closed.

(d) The second implication follows directly from the unitarity calculation performed above. The first one is connected deeply to the *Jarlskog invariant*

$$J = \mathrm{Im}\left[V_{ud}V_{ub}^{*}V_{cd}^{*}V_{cb}\right] \,, \tag{3.3}$$

encoding the amount of CPV in the theory. Find the proportionality factor between J and the UT's area.