II.3 Electrodynamics

Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \rho \qquad \vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$\vec{E}, \vec{B}$ can be expressed through 4-potential

$$A^\mu = (\phi, \vec{A}) \quad \text{with} \quad \partial^\mu = \begin{pmatrix} \partial_t \\ -\vec{\nabla} \end{pmatrix}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}, \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$B^i = -\varepsilon_{ijk}\,\partial^j A^k \qquad E^i = \partial^i A^0 - \partial^0 A^i$$

e.g.: $B^1 = -\partial^2 A^3 + \partial^3 A^2$

$\rightarrow$ fields can be collected in electromagnetic tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

(antisymmetric: $F^{\mu\nu} = -F^{\nu\mu}$)

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$\Rightarrow$ The inhomogeneous Maxwell's equations can then be written as

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu \quad , \quad \text{wit} \quad j^\nu = \begin{pmatrix} \rho \\ \vec{j} \end{pmatrix}}$$

homogeneous Maxwell's equations:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad , \quad \text{with} \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$\rightarrow \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \partial_\rho A_\sigma = 0$$

$\rightarrow$ always satisfied if A_σ are regular functions.

Gauge invariance

$F^{\mu\nu}$ is invariant under gauge transformations

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \lambda(x)$$

$$F^{\mu\nu} \to F'^{\mu\nu} = \partial^\mu (A^\nu + \partial^\nu \lambda) - \partial^\nu (A^\mu + \partial^\mu \lambda)$$

$$= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$

$\Rightarrow$ redundancies when writing electromagnetic field using 4-potential A^μ

4 degrees of freedom, but only
2 physical d.o.f. of EM waves / photons

(more details: sect. II.5)

To obtain the correct number of d.o.f., we can fix the gauge with additional gauge conditions.

Typical choices:

Lorenz gauge: $\partial_\mu A^\mu = 0$

Ludwig Lorenz

→ Lorentz invariant

Hendrik Lorentz

removes only 1 d.o.f.

Radiation / Coulomb gauge: $A^0 = 0$, $\vec{\nabla} \cdot \vec{A} = 0$

→ removes 2 d.o.f.,
implies Lorenz gauge
breaks Lorentz invariance

In Lorenz gauge, Maxwell's equations simplify:

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}_{=0} = \Box A^\nu = j^\nu$$

For $j^\nu = 0$ (i.e. without sources) we obtain a massless Klein-Gordon eq. for each ν:

$$\Box A^\nu = 0$$

$\Rightarrow$ plane wave solutions

$$A^\nu \sim \varepsilon^\nu e^{-ikx} + \text{c.c.}, \qquad k^2 = 0$$

with polarization vector $\varepsilon^\nu = \begin{pmatrix} \varepsilon_0 \\ \vec{\varepsilon} \end{pmatrix}$

In Coulomb gauge: $\varepsilon_0 = 0, \quad \vec{\varepsilon} \cdot \vec{k} = 0$

$\rightarrow$ polarizations transverse to momentum $\vec{k}$

Lagrangian density of the el.-magn. field

coupling to external current j^μ

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

$$= -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j_\mu A^\mu$$

$$= -\frac{1}{2} \left[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\nu A_\mu)(\partial^\mu A^\nu) \right] - j_\mu A^\mu$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial(A_\nu)} = -j^\nu$$

$\Rightarrow$ Euler - Lagrange eq.

$$\partial_\mu F^{\mu\nu} = j^\nu$$

One possible source of the EM field can be the current of a Dirac field ψ with charge e

$$j^\mu = e\bar{\psi}\gamma^\mu\psi$$

$\Rightarrow$ Lagrangian density of Quantum electro-dynamics

$$\mathcal{L}_{QED} = \bar{\psi}i\not{\partial}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\psi$$
$$= \bar{\psi}\left[(i\partial_\mu - eA_\mu)\gamma^\mu - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

with the covariant derivative

$$\boxed{D_\mu = \partial_\mu + ie A_\mu}$$

$\Rightarrow$ This corresponds to minimal coupling in classical electrodynamics via the substitution

$$p^\mu \rightarrow p^\mu - e A^\mu$$

local U(1) invariance

We consider the variation of $\mathcal{L}_{QED}$ under the simultaneous transforms

$$\psi \rightarrow \psi' = e^{-ie\lambda(x)} \psi$$

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \lambda(x) \leftarrow \text{gauge transform}$$

Terms of $\mathcal{L}_{QED}$:

- $F_{\mu\nu} F^{\mu\nu}$ is invariant under these transformations

- For the Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi$$

of the free Dirac field, we found

$\alpha(x) = e\lambda(x)$

$$\delta\mathcal{L} = \frac{\partial\alpha}{\partial x^\mu}\bar{\Psi}\gamma^\mu\Psi$$

$$= e\,(\partial_\mu\lambda)\,\bar{\Psi}\gamma^\mu\Psi$$

In QED, this term is cancelled by the transformation of the

- interaction term

$$-e A_\mu \bar{\Psi}\gamma^\mu\Psi \to -e(A_\mu + \partial_\mu\lambda)\bar{\Psi}\gamma^\mu\Psi$$

$\Rightarrow \delta\mathcal{L}_{QED} = 0$

$\Rightarrow$ The gauge freedom of the EM field results in a local $U(1)$ symmetry of QED.

Similarly, local $SU(2)$ and $SU(3)$ symmetries will lead to the weak & strong interactions.

II.4. Lie groups

A **group** is a set $G = \{g_i\}$ together with an operation $\circ$, with the properties

- $g_1, g_2 \in G \Rightarrow g_1 \circ g_2 \in G$
- $\exists\, e \in G$ with $e \circ g = g \circ e = g \quad \forall g \in G$
- $\forall g \in G \quad \exists\, g^{-1}$ with $g \cdot g^{-1} = g^{-1} g = e$
- $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

If in addition

$$g_1 \circ g_2 = g_2 \circ g_1 \qquad \forall\, g_1, g_2 \in G$$

the group is called **abelian**.

A **representation** R maps each group element g to a $n \times n$ matrix $D_R(g)$

n: dimension of repr. R

$$g \to D_R(g)$$

with $D_R(e) = \mathbb{1}$

$$D_R(g_1 \circ g_2) = D_R(g_1) \cdot D_R(g_2)$$

→ Each group element „represented" by a matrix, such that matrix multiplication preserves group properties.

Two representations D_1 and D_2 are equivalent if there exists a matrix A, such that

$$D_2(g) = A^{-1} D_1(g) A \quad \forall g \in G$$

→ this corresponds to a basis change

A representation is reducible if it is equivalent to a representation in block-diagonal form:

$$D(g) \to A^{-1} D(g) A = \begin{pmatrix} D^{(1)}(g) & 0 \\ 0 & D^{(2)}(g) \end{pmatrix}$$

$\underbrace{\qquad}_{k \text{ components}}$ $\underbrace{\qquad}_{n-k \text{ compon.}}$

with matrices $D^{(1)}(g)$, $D^{(2)}(g)$

If a representation is not reducible, it is called irreducible.

example: $\vec{S} = \vec{S}_1 + \vec{S}_2$ acting on 2 spin-$\frac{1}{2}$ particles

$\vec{S}\,|s_1 s_2\rangle$, S_i are not block diagonal in basis $(|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle)$

basis change $|s_1, s_2\rangle \to |s, m_s\rangle$

$$\to \quad S_i = \begin{pmatrix} S_i^{(1)} & 0 \\ 0 & S_i^{(0)} \end{pmatrix}$$

with 3x3 matrices $S_i^{(1)}$ acting on triplet states $(|1,1\rangle, |1,0\rangle, |1,-1\rangle)$,

1x1 matrix $S_i^{(0)}$ acting on singlet state $|0,0\rangle$

$\to \vec{S}$ is reducible, can be decomposed into 2 irreducible representations $S^{(1)}$, $S^{(0)}$, acting on the subgroups with $s=1$, $s=0$, respectively.

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$