A Lie group $G = \{ U(\theta_1 \ldots \theta_N) \mid \theta_i \in \mathbb{R} \}$ is an infinite group with elements specified by finite number of real parameters $\theta_1 \ldots \theta_N$ in a continuous and differentiable way, with $U(0) = 1$

Example: rotations in $\mathbb{R}^3$

$\rightarrow$ 3 angles $\theta_x, \theta_y, \theta_z$

e.g. $$R_z(\theta_z) = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Generators of Lie group

$$T^a = \frac{1}{i} \frac{\partial U}{\partial \theta^a} \Big|_{\vec{\theta}=0}$$

$\rightarrow$ specifics change under infinitesimal transform $\delta\vec{\theta}$

$$U(\delta\vec{\theta}) = \mathbb{1} + i \sum_a \delta\theta^a \cdot T^a + \mathcal{O}((\delta\vec{\theta})^2)$$

comments: • T^a don't depend on $\vec{\theta}$
• T^a depend on representation
$\rightarrow T^a_R = \frac{1}{i} \frac{\partial}{\partial \theta^a} D_R(U(\theta))$

finite transform $U(\vec{\theta})$:

$N \to \infty$ limit of infinitesimal transforms $\delta\vec{\theta} = \frac{\vec{\theta}}{N}$

$$U(\vec{\theta}) = \lim_{N\to\infty} \left(\mathbb{1} + i \frac{\theta^a T^a}{N} \right)^N = e^{i\theta^a T^a}$$

$\rightarrow$ each group element can be written as

$$U(\vec{\theta}) = e^{i\vec{\theta}\vec{T}}$$

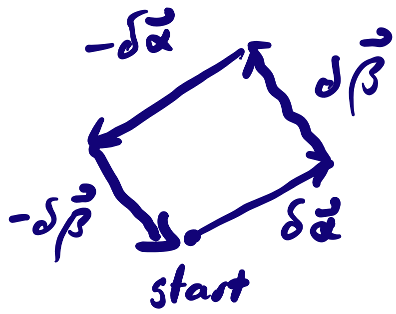
Is $U(\delta\vec{\theta})$ unitary?

$$\begin{aligned} \mathbb{1} &= U^\dagger(\delta\vec{\theta})\, U(\delta\vec{\theta}) \\ &= (\mathbb{1} - i\,\delta\theta^a T^{a\dagger})(\mathbb{1} + i\,\delta\theta^a T^a) \\ &= \mathbb{1} + i\,\delta\theta^a (T^a - T^{a\dagger}) \end{aligned}$$

$\Rightarrow \quad U^\dagger = U^{-1}$ if $T^{a\dagger} = T^a$

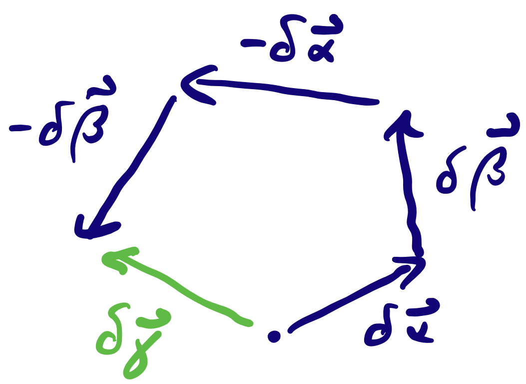
$\rightarrow T^a$ hermitean

"Undoing" transformations $U(\delta\vec{\beta})\,U(\delta\vec{\alpha})$:



not true in general

$\rightarrow$ need to consider each transformation in local system, depending on previous transformations:



$$U(\delta\vec{\gamma}) = \left(\mathbb{1} + i\delta\gamma^a T^a + \frac{1}{2}(i\gamma^a T^a)^2 + \dots\right)$$

$$= \left(\mathbb{1} + i\delta\gamma^a T^a - \frac{1}{2}\delta\gamma_a\,\delta\gamma_b\,T^a T^b + \dots\right)$$

$$= U(-\delta\vec{\beta})\,U(-\delta\vec{\alpha})\,U(\delta\vec{\beta})\,U(\delta\vec{\alpha})$$

$$= \left(\mathbb{1} - i\,\delta\beta^a T^a - \frac{1}{2}\delta\beta^a\delta\beta^b T^a T^b + \dots\right)$$
$$\cdot\left(\mathbb{1} - i\,\delta\alpha^a T^a - \frac{1}{2}\delta\alpha^a\delta\alpha^b T^a T^b + \dots\right)$$
$$\cdot\left(\mathbb{1} + i\,\delta\beta^a T^a - \frac{1}{2}\delta\beta^a\delta\beta^b T^a T^b + \dots\right)$$
$$\cdot\left(\mathbb{1} + i\,\delta\alpha^a T^a - \frac{1}{2}\delta\alpha^a\delta\alpha^b T^a T^b + \dots\right)$$

terms linear in $\delta\alpha$, $\delta\beta$ cancel

terms $\delta\alpha\cdot\delta\alpha$, $\delta\beta\delta\beta$ cancel

$$= \mathbb{1} - \delta\beta^a\delta\alpha^b T^a T^b + \delta\beta^a\delta\alpha^b T^a T^b$$
$$+ \delta\alpha^a\delta\beta^b T^a T^b - \underbrace{\delta\beta^a\delta\alpha^b T^a T^b}_{\delta\beta^b\delta\alpha^a T^b T^a}$$

$$= \mathbb{1} + \delta\alpha^a\delta\beta^b\left(T^a T^b - T^b T^a\right)$$

$$\Rightarrow \quad i\delta\gamma^c T^c = \delta\alpha^a\delta\beta^b\left(T^a T^b - T^b T^a\right)$$
$$= \delta\alpha^a\delta\beta^b\,[T^a, T^b]$$

$\delta\gamma^c \to 0$ if $\delta\alpha^a \to 0$ or $\delta\beta_b \to 0$

$$\Rightarrow \quad \delta\gamma^c = f^{abc}\cdot\delta\alpha^a\delta\beta^b$$

with f^{abc} independent of $\delta\vec{\alpha}, \delta\vec{\beta}$

$$\Rightarrow \quad [T^a, T^b] = i f^{abc} \, T^c$$

$\Rightarrow$ Lie Algebra defined by commutator of generators, specified by structure constants f^{abc},

$$f^{abc} = - f^{bac}$$

The structure constants are independent of the representation

$\Rightarrow$ finding all group representations R amounts to finding all matrix solutions T_R^a satisfying the Lie algebra.

A representation R can be characterized by their Casimir invariants C_λ^R wit

- $[C_\lambda^R, T_R^a] = 0 \quad \forall a$
- C_λ^R polynomial in T_R

$$\rightarrow \quad c_X^R = \sum_i c_i T^i + \sum_{ij} c_{ij} \cdot T^i T^j + \dots$$

Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0$$

(valid for arbitrary square matrices T^a, T^b, T^c

$$\Rightarrow [T^a, if^{bck} T^k] + [T^b, if^{cal} T^l] + [T^c, if^{abm} T^m]$$

$$= if^{akn} if^{bck} T^n + if^{bln} \underbrace{if^{cal}}_{-f^{acl}} T^n + if^{cmn} if^{abm} T^n$$

$$= \left[(-if^{bck})(-if^{akn}) - (-if^{acl})(-if^{bln}) + if^{abm}(-if^{mcn})\right] \cdot T^n$$

define matrices $(F^a)_{ij} = -if^{aij}$

$$= \left[F^b F^a - F^a F^b + if^{abm} \cdot F^m\right]_{cn} T^n = 0$$

$$\Rightarrow [F^a, F^b] = if^{abm} F^m$$

$\Rightarrow$ The matrices $(F^a)_{ij} = -i f^{aij}$ are a representation of the group. This is the adjoint representation.

examples

- $SO(3) \to$ rotations in $\mathbb{R}^3$

  generators L_x, L_y, L_z with algebra

  $$[L_i, L_j] = i\varepsilon_{ijk} L_k$$

  $\to$ algebra of angular momentum

  Casimir invariant $\vec{L}^2 = \sum_i L_i L_i$

  $$[L_i, \vec{L}^2] = 0$$

  $\to$ can diagonalize $\vec{L}^2$ and L_z simultaneously

  $\to$ states $|\ell, m\rangle$ with

  - $\vec{L}^2 |\ell, m\rangle = \ell(\ell+1) |\ell, m\rangle$
  - $L_z |\ell, m\rangle = m |\ell, m\rangle$
  - $L_x, L_y \; |\ell, m\rangle = \sum_{m'} c_{m'} |\ell, m'\rangle$

→ irreducible representation for fixed values of ℓ, acting on $(2\ell+1)$-dimensional multiplets

$$|\ell,-\ell\rangle \;\dots\; |\ell,\ell\rangle$$

- $SU(N)$: $N\times N$ matrices U with

$$U^\dagger U = \mathbb{1}\,, \qquad \det U = 1$$

Generators T^a (in fundamental represent.)

- T^a hermitean $N\times N$ matrices

  → N^2 real parameters

- T^a are trace-less, such that $\det U = 1$:

$$\det U = \det e^{i\theta^a T^a} = e^{\mathrm{tr}(i\theta^a T^a)} = 1$$

$$\det U = \det(\underbrace{A U A^{-1}}_{\text{diagonal}})$$

$\Rightarrow$ $SU(N)$ has N^2-1 generators

For SU(2): can choose generators

$$T^i = \frac{1}{2} \sigma^i$$

with Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[T^i, T^j] = i\varepsilon_{ijk} T^k$$

$\rightarrow$ Lie algebra identical to algebra of SO(3)

For SU(3): 8 generators $T^a = \frac{1}{2} \lambda^a$

with Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

↖ normalization chosen such that $\mathrm{tr}(\lambda^a \lambda^b) = 2\,\delta^{ab}$

Lie algebra of $SU(3)$:

$$[T^a, T^b] = i f^{abc}\, T^c$$

with f^{abc} completely anti-symmetric,

$$f^{123} = 1, \quad f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

comment:

T^3 and T^8 commute with all T^i

→ can define Clebsch-Gordan coefficients for products of $SU(3)$ representations analoguous to angular momentum addition

→ hadron multiplets,
hadrons specified by Casimir invariants
and eigenvalues of T^3, T^8.

representations of $SU(3)$:

- trivial repr. → $U(\vec{\theta}) = \mathbb{1} \quad \forall \vec{\theta}$
- fundamental repr: → $T_F^a = \frac{\lambda^a}{2}$
- anti-fundamental repr.: $T_{F^*}^a = -\frac{\lambda^{a*}}{2}$
- adjoint repr: $T_A^a = (F^a)_{bc} = -if^{abc}$

→ 8×8 matrices

quadratic Casimir invariants

$$C_F = \sum_a T_F^a T_F^a = \frac{N^2-1}{2N} = \frac{4}{3}$$

$$C_A = \sum_a F^a F^a = N = 3$$

→ these invariants will appear as color factors in QCD