II.5 Non-abelian gauge theories

We previously found the QED Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $D_\mu = \partial_\mu + ieA_\mu$

is invariant under the local $U(1)$ transformation

$$\psi \to e^{-ie\lambda(x)} \psi$$

$$A_\mu \to A_\mu + \partial_\mu \lambda(x)$$

We want to find a Lagrangian with a local $SU(N)$ symmetry:

- starting point: Lagrangian density of N identical Dirac fields ψ_i

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i (i\not{\partial} - m)\psi_i = \bar{\psi}(i\not{\partial} - m)\psi$$

with $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$

→ invariant under global transform

$$\psi_i \longrightarrow U_{ij}\,\psi_j \,,$$

$$U = e^{i\theta^a T^a} \,, \qquad \theta^a = \text{const.}$$

T^a: $SU(N)$ generators in fundamental repr.

- local transform: $U(x) = e^{ig\,\theta^a(x)\, T^a}$

$\rightarrow$ try to identify covariant derivative

$$D_\mu = \partial_\mu - ig A_\mu = \partial_\mu - ig\, A_\mu^a\, T^a$$

with (N^2-1) gauge fields A_μ^a such that

$$\mathcal{L} = \bar\psi(i\not{D} - m)\psi = \bar\psi_i\,(i\not{D}_{ij} - m\,\delta_{ij})\,\psi_j$$

is invariant under transform $U(x)$

$$\mathcal{L} \rightarrow \mathcal{L}' = \bar\psi U^{-1}(i\not{D}' - m)\, U\psi$$

$\Rightarrow$ we need $D'_\mu \underbrace{(U\psi)}_{\psi'} = U\,(D_\mu \psi)$

$$\Rightarrow \left(\partial_\mu - ig A'_\mu\right) U\psi$$

$$= (\partial_\mu U)\psi + U(\partial_\mu \psi) - ig\, A'_\mu U \psi$$

$$\overset{!}{=} U D_\mu \psi = U(\partial_\mu \psi) - U ig\, A_\mu \psi$$

$$\Rightarrow \quad (\partial_\mu U) - ig A'_\mu U = - U ig\, A_\mu$$

$$\Rightarrow \quad A'_\mu = U A_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}$$

infinitesimal transform:

$$A'_\mu = A_\mu + ig\theta^b \left(T^b A_\mu - A_\mu T^b \right) - \frac{i}{g} ig \partial_\mu \theta^b T^b$$

$$A'^a_\mu T^a = A^a_\mu T^a + ig\theta^b\, if^{bca} A^c_\mu T^a + (\partial_\mu \theta^a) T^a$$

$$\rightarrow \quad A^a_\mu \rightarrow A'^a_\mu = A^a_\nu - g f^{abc} \theta^b A^c_\mu + \partial_\mu \theta^a$$

Comments:

- number of gauge fields
  = number of generators $= N^2 - 1$ for $SU(N)$

$SU(2)$: 3 gauge fields

$\rightarrow W^+, W^-, Z$ bosons of weak interaction

SU(3): 8 gauge fields

$\rightarrow$ 8 gluons in QCD

- We could have started with fields ψ_i in different representation of $SU(N)$, transformation of gauge fields independent of representation!

We can add a gauge invariant kinetic term of gauge fields to the Lagrangian

$$\mathcal{L}_{gauge} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$\Rightarrow \mathcal{L}_{YM} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$

Yang-Mills Lagrangian

The terms $gf^{abc}A_\mu^b A_\nu^c$ in the Lagrangian only apper due to the non-abelian group structure and will lead to self-interactions of the gauge fields.

II.6 The Lorentz & Poincaré group

Lorentz transformations

linear coordinate transforms Λ,

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

which leave

$$x^2 = g_{\mu\nu} x^\mu x^\nu = \underbrace{x_0^2}_{=t^2} - \vec{x}^2$$

invariant. (Lorentz group = $O(3,1)$)

$$\Rightarrow \quad g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma = g_{\rho\sigma} x^\rho x^\sigma$$

$$\Rightarrow \quad \Lambda^\mu{}_\rho \, g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad \Rightarrow \quad \boxed{\Lambda^T g \Lambda = g}$$

This in particular implies

- $\det \Lambda = \pm 1$, since

$$\det g = \det(\Lambda^T g \Lambda) = (\det \Lambda)^2 \cdot \det g$$

$$\Rightarrow \quad (\det \Lambda)^2 = 1$$

- $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$

since for $\rho = \sigma =$

$$\Lambda^\mu{}_0 \cdot g_{\mu\nu} \Lambda^\nu{}_0 = g_{00}$$

$$(\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = 1$$

$$\Rightarrow (\Lambda^0{}_0)^2 \geq 1$$

types of Lorentz transformations

- rotations $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ with $R^T R = \mathbb{1}_3$

$\rightarrow$ parametrized by 3 angles

- boosts,

e.g. in x direction:

$$t \rightarrow t' = \gamma(t + vx)$$
$$x \rightarrow x' = \gamma(x + vt)$$

$(-1 \leq v \leq 1,$
$\gamma = 1/\sqrt{1-v^2})$

$$\Lambda = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with the rapidity η, $v = \tanh\eta$

- disrete transformations

parity $\quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

time reversal $\quad T = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

classification of Lorentz transforms:

$\det\Lambda$ / $\Lambda^0{}_0$	$+1$	-1
≥ 1	$\mathcal{L}_+^\uparrow$	$\mathcal{L}_-^\uparrow$
≤ -1	$\mathcal{L}_+^\downarrow$	$\mathcal{L}_-^\downarrow$

← orthochronous transforms

↑ proper transforms $\quad (SO(3,1))$

subgroups:
(→ closed)

- $\mathcal{L}_+^\uparrow$
- $\{\mathbb{1}, P\}$
- $\{\mathbb{1}, T\}$

$$\Rightarrow \quad \mathcal{L} = \mathcal{L}_+^\uparrow \otimes \underbrace{\{\mathbb{1}, P\} \otimes \{\mathbb{1}, T\}}_{}$$

important for weak interaction/
flavor physics

The proper, orthochronous Lorentz group

($SO^+(3,1)$)

→ boosts, rotations

→ Lie group with 6 parameters
- angles $\vec{\alpha}$
- rapidity $\vec{\eta}$

Infinitesimal transforms:

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \delta\omega^\mu{}_\nu + \mathcal{O}\left((\delta\omega)^2\right)$$

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \overset{!}{=} g_{\rho\sigma}$$

$$= g_{\mu\nu} \left(\delta^\mu_\rho + \delta\omega^\mu{}_\rho\right)\left(\delta^\nu_\sigma + \delta\omega^\nu{}_\sigma\right)$$

$$= g_{\rho\sigma} + \delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma}$$

$\rightarrow \quad \delta\omega_{\sigma\rho} = -\delta\omega_{\rho\sigma}$ antisymmetric

$\rightarrow$ 6 parameters

$\delta\omega_{0i} = -\delta\omega_{i0}$ $\quad i = 1 \ldots 3 \rightarrow$ 3 boosts

$\delta\omega_{ij} = -\delta\omega_{ji}$ $\quad i,j = 1 \ldots 3$, $i \neq j$ $\rightarrow$ 3 rotations

finite rotations:

$$\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\alpha\beta} M^{\alpha\beta}}\right)^\mu{}_\nu$$

with generators $\left(M^{\alpha\beta}\right)^\mu{}_\nu = i\left(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu\right)$

such that $\Lambda^\mu{}_\nu = \delta^\mu_\nu + d\omega^\mu{}_\nu = \delta^\mu_\nu + \delta\omega_{\alpha\beta}\, g^{\alpha\mu}\delta^\beta_\nu$

$$\stackrel{!}{=} \delta^\mu_\nu - \frac{i}{2}\delta\omega_{\alpha\beta}\left(M^{\alpha\beta}\right)^\mu{}_\nu$$

This leads to the Lorentz algebra

$$[M^{\alpha\beta}, M^{\gamma\delta}] = i\left(g^{\beta\gamma}M^{\alpha\delta} - g^{\alpha\gamma}M^{\beta\delta} - g^{\beta\delta}M^{\alpha\gamma} + g^{\alpha\delta}M^{\beta\gamma}\right)$$

Alternatively, this can be written with the generators of

- boost $\quad K^i = M^{i0}$
- rotations $\quad J^i = \frac{1}{2}\varepsilon^{ijk} \cdot M^{jk}$

$$\Rightarrow \quad [J^i, J^j] = i\varepsilon^{ijk} J^k$$

← J^i are a subgroup

$$[J^i, K^j] = i\varepsilon^{ijk} K^k$$

$$[K^i, K^j] = -i\varepsilon^{ijk} J^k$$

boost × boost = transform involving rotations

The algebra simplifies if we define

$$N^{\pm, i} = \frac{1}{2}\left(J^i \pm i K^i\right)$$

$$\Rightarrow [N^{+i}, N^{+j}] = i\varepsilon^{ijk} N^{+k}$$

$$[N^{-i}, N^{-j}] = i\varepsilon^{ijk} N^{-k}$$

$$[N^{+i}, N^{-j}] = 0$$

$\Rightarrow$ The Lorentz algebra (not group!) can be written as the algebra of $SU(2) \otimes SU(2)$

$\rightarrow$ The fields / particles can be classified by the representation of $SU(2) \otimes SU(2)$, under which they transform

repr (n^-, n^+)	number components	name	transformations
$(0,0)$	1	scalar ϕ	$\phi' = \phi$
$(\frac{1}{2}, 0)$	2	left-handed Weyl spinor ψ_L	$\psi_L' = \Lambda_L \psi_L$ $\Lambda_L = e^{(-i\vec{\alpha} - \vec{\eta})\frac{\vec{\sigma}}{2}}$
$(0, \frac{1}{2})$	2	right-handed Weyl spinor ψ_R	$\psi_R' = \Lambda_R \psi_R$ $\Lambda_R = e^{(-i\vec{\alpha} + \vec{\eta})\frac{\vec{\sigma}}{2}}$
$(\frac{1}{2},0) \oplus (0,\frac{1}{2})$	4	Dirac spinor $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$	
$(\frac{1}{2}, \frac{1}{2})$	4	Vectors V^μ	$V'^\mu = \Lambda^\mu{}_\nu V^\nu$
$(1,0)$			
$\vdots$			