Lecture 5:

Lorentz algebra can be written as

$$SU(2) \otimes SU(2)$$

repr (n^-, n^+)	number components	name	transformations
$(0,0)$	1	scalar ϕ	$\phi' = \phi$
$(\frac{1}{2}, 0)$	2	left-handed Weyl spinor ψ_L	$\psi_L' = \Lambda_L \psi_L$ $\Lambda_L = e^{(-i\vec{\alpha} - \vec{\eta})\frac{\vec{\sigma}}{2}}$
$(0, \frac{1}{2})$	2	right-handed Weyl spinor ψ_R	$\psi_R' = \Lambda_R \psi_R$ $\Lambda_R = e^{(-i\vec{\alpha} + \vec{\eta})\frac{\vec{\sigma}}{2}}$
$(\frac{1}{2},0) \oplus (0,\frac{1}{2})$	4	Dirac spinor $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$	
$(\frac{1}{2}, \frac{1}{2})$	4	Vectors V^μ	$V'^\mu = \Lambda^\mu{}_\nu V^\nu$
$(1,0)$			
⋮			

In the Lagrangian density, only combinations of fields resulting in Lorentz scalars are allowed, e.g.:

- $\psi_R^\dagger \psi_L$ $\quad \left(\psi_R^\dagger \underbrace{\Lambda_R^\dagger \Lambda_L}_{=\mathbb{1}} \psi_L = \psi_R^\dagger \psi_L \right)$

  We can't write down mass terms for Weyl spinors, since $m^2 \psi_L^\dagger \psi_L$ not Lorentz invariant

- $\bar{\psi} \psi = \psi^\dagger \gamma_0 \psi = \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R$

  $\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ in chiral repr.

- $\partial_\mu \bar{\psi} \gamma^\mu \psi$

Additional requirements:

- Mass dimension $[\mathcal{L}] = 4$

  such that $[S] = [\int d^4x \, \mathcal{L}] = 0$

We can assign:

$$[m] = [\partial_\mu] = 1$$

$$[\phi] = 1$$

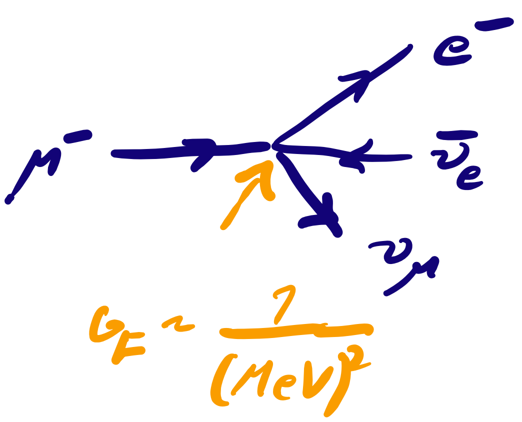
$$[\psi] = \frac{3}{2}$$

$$[A] = 1$$

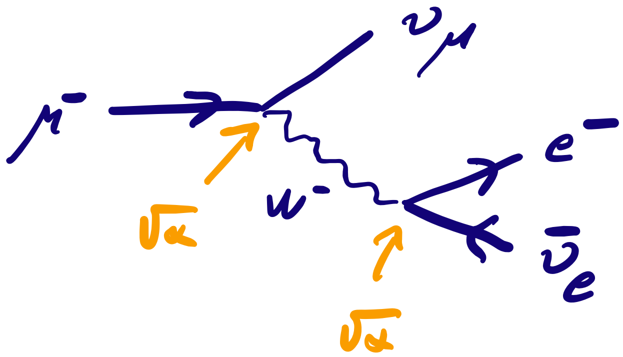
- for fully consistent theory we need $[g] \geq 0$ for all coupling constants g_i

but $[g] < 0$ can appear in approximations of full theory. $\rightarrow$ Effective Field Theories (EFT)

example EFT: $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$



is low energy approximation to



The Poincaré group

The Lorentz group can be extended by translations a^μ $\rightarrow$ Poincaré group

$$x'^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$\rightarrow$ 10 parameters:
- 3 boosts
- 3 rotations
- 4 translationen

As generator for translations, we can identify the operator $P^\mu = i\partial^\mu$, leading to the Poincaré algebra

$$[P^\alpha, P^\beta] = 0$$

$$[P^\alpha, M^{\beta\gamma}] = i(g^{\alpha\beta} P^\gamma - g^{\alpha\gamma} P^\beta)$$

$$[M^{\alpha\beta}, M^{\gamma\delta}] = \ldots \qquad \text{(as before)}$$

Casimir invariants of the Poincaré group

- $P_\mu P^\mu$

proof: $[P_\mu P^\mu, P^\nu] = 2 P_\mu [P^\mu, P^\nu] = 0$

$$[P_\mu P^\mu, M^{\alpha\beta}] = 2 P_\mu [P^\mu, M^{\alpha\beta}]$$

$$= 2 P_\mu \cdot i (g^{\mu\alpha} P^\beta - g^{\mu\beta} P^\alpha)$$

$$= 0$$

- $W_\mu W^\mu$ with the Pauli-Lubanski vektor

$$\left[\begin{array}{l} \varepsilon_{0123} = 1 \\ \varepsilon^{0123} = -1 \end{array}\right]$$

$$W^\mu = -\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma$$

proof: • $W_\mu W^\mu$ is Lorentz invariant

$$\bullet\ [W^\mu, P^\alpha] = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\underbrace{[M_{\nu\rho}, P^\alpha]}_{=i(g^\alpha_\nu P_\rho - g^\alpha_\rho P_\nu)} P_\sigma$$

$$= -\varepsilon^{\mu\alpha\rho\sigma}P_\rho P_\sigma = 0$$

$$\underbrace{\varepsilon^{\mu\alpha\rho\sigma}}_{\text{anti-symm}}\ \underbrace{P_\rho P_\sigma}_{\text{symm.}}$$

under $\rho \leftrightarrow \sigma$

We can apply these Casimir invariants to 1-particle states $|p, s\rangle$ with momentum p and spin s:

- $P_\mu P^\mu |p,s\rangle = p^2 |p,s\rangle = m^2 |p,s\rangle$

- $W_\mu W^\mu |p,s\rangle$

<u>for $m \neq 0$</u>: choose rest frame $p^\mu = (m, \vec{0})$

$$\Rightarrow W^0 = 0 \quad , \quad W^i = \frac{m}{2}\varepsilon^{0ijk} M_{jk} = m J^i$$

$$\Rightarrow -W_\mu W^\mu |p,s\rangle = m^2 s(s+1) |p,s\rangle$$

$\rightarrow$ $(2s+1)$ spin degrees of freedom for massive particles

for $m=0$

one can show that $W^\mu = h \cdot p^\mu$ with the helicity

$$h = \frac{\vec{p}}{|\vec{p}|} \cdot \vec{J} \quad , \quad h \to \pm s$$

$\rightarrow$ 2 spin degrees of freedom for massless particles

$\Rightarrow$ Classification of elementary particles according to

- transformation under Lorentz transforms: scalars, Weyl- / Dirac- spinors, vectors
- mass m
- spin degrees of freedom:

- 2 for $m = 0$
- $2s+1$ for $m \neq 0$

III Quantization

here only general ideas
detailed discussion: TTP 1

III.1 canonical quantization, Fock space

reminder: starting from classical point particles described by $S = \int dt \, L(x_i, \dot{x}_i)$,

a quantized description can be obtained by interpreting x_i and $p_i = \frac{\partial L}{\partial \dot{x}_i}$ as operators acting on a Hilbert space, with the canonical commutation relation

$$[x_i, p_j] = i\hbar \delta_{ij} = i \delta_{ij}$$

The dynamics described by Hamiltonian

$$H = \sum_i \dot{x}_i p_i - L$$

In Quantum Field Theory,
we interpret fields $\phi(x)$, $\psi(x)$... as operators with position x as continuous index.

We then impose canonical quantization conditions on the fields $\phi_i(x)$ and conjugate momenta $\Pi_i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)}$.

The excitations of the field can then be interpreted as particles.

Quantization of the real Klein-Gordon field

Starting from the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

we can derive the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0$$

and we can obtain the (positive definite) Hamiltonian density

$$\mathcal{H} = \Pi \cdot \partial_0 \phi - \mathcal{L}$$

$$= \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2$$

where $\Pi(x) = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$

is the conjugate momentum of the field $\phi(x)$.

The solutions of the Klein-Gordon equation are plane waves with the Fourier decomposition

$$\boxed{\phi(x) = \underbrace{\int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p}}_{=:\ \int d\tilde{p}} \left[a(\vec{p})\, e^{-ipx} + a^\dagger(\vec{p})\, e^{+ipx} \right] \quad \text{with } E_p = +\sqrt{\vec{p}^{\,2} + m^2}}$$

note: • Fourier coefficients $a(\vec{p})$, $a^+(\vec{p})$ will be treated as operators in the following, chosen such that $\phi^+(x) = \phi(x)$

• $\int d\tilde{p} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \overset{(*)}{=} \int \frac{d^4 p}{(2\pi)^4} 2\pi\delta(p^2-m^2)\Theta(p_0)$

is the Lorentz invariant phase space measure

(*) $\delta(p^2-m^2) = \delta(E^2-(\vec{p}^2+m^2))$

$= \delta((E-\sqrt{\vec{p}^2+m^2})(E+\sqrt{\vec{p}^2+m^2}))$

$= \frac{1}{2E_p}[\delta(E-E_p) + \underbrace{\delta(E+E_p)}_{\to E<0}]$

$\delta(f(x)) = \sum_{\substack{\text{zeros} \\ f(x_i)=0}} \frac{1}{|f'(x_i)|}\delta(x-x_i)$

We now impose canonical quantization conditions at fixed time t,

$$[\phi(\vec{x},t),\ \pi(\vec{x}',t)] = i\,\delta^3(\vec{x}-\vec{x}')$$

$$[\phi(\vec{x},t),\ \phi(x',t)] = [\pi(\vec{x},t),\ \pi(\vec{x}',t)] = 0$$

This leads to

$$[a(\vec{p}),\ a^+(\vec{p}')] = (2\pi)^3\, 2E_p\ \delta^3(\vec{p}-\vec{p}')$$

$$[a(\vec{p}),\ a(\vec{p}')] = [a^+(\vec{p}),\ a^+(\vec{p}')] = 0$$

in analogy to the harmonic oscillator in QM.

$\rightarrow$ The field can be seen as superposition of harmonic oscillators and we can interpret

$$n(\vec{p}) = a^+(\vec{p})\ a(\vec{p})$$

as particle density operator, and

$$N = \int d\tilde{p} \quad n(\vec{p})$$

as particle number operator.