Lecture 6:

Quantization of real scalar field

$$\phi(x) = \underbrace{\int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p}}_{=:\int d\tilde{p}} \left[a(\vec{p})\, e^{-ipx} + a^\dagger(\vec{p})\, e^{+ipx} \right]$$

with $E_p = +\sqrt{\vec{p}^{\,2} + m^2}$

$$[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\,\delta^3(\vec{x}-\vec{x}')$$

$$[\phi(\vec{x},t), \phi(x',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

$$\Rightarrow [a(\vec{p}), a^\dagger(\vec{p}')] = (2\pi)^3\, 2E_p\, \delta^3(\vec{p}-\vec{p}')$$

$$[a(\vec{p}), a(\vec{p}')] = [a^\dagger(\vec{p}), a^\dagger(\vec{p}')] = 0$$

<u>particle density operator</u>

$$n(\vec{p}) = a^\dagger(\vec{p})\, a(\vec{p})$$

<u>particle number operator.</u>

$$N = \int d\tilde{p}\; n(\vec{p})$$

The energy and momentum of the fields were defined as (→ sect. II.2 symmetries)

$$H = \int d^3\vec{x} \; \mathcal{H} = \int d^3\vec{x} \; T^0{}_0 = \int d^3\vec{x} \left(\Pi \cdot \partial_0 \phi - \mathcal{L} \right)$$

$$P_i = \int d^3\vec{x} \; T^0{}_i = \int d^3\vec{x} \left(\Pi \cdot \partial_i \phi \right)$$

Inserting the Fourier decomposition of ϕ leads to

$$H = \int d\tilde{p} \; E_p \left(n(\vec{p}) + c \right)$$

$$P_i = \int d\tilde{p} \; \vec{p} \left(n(\vec{p}) + c \right)$$

$c = \text{const} \;\hat{=}\;$ ground state energy

$\Rightarrow \int d\tilde{p} \; c = \infty$, but only energy differences are relevant

→ can be ignored → set $c=0$

formal treatment: normal order (TTP 1)

$\Rightarrow$ Energy & momentum of the field are given by the sum of the excitations.

$\Rightarrow$ We also find $\langle H \rangle = \langle \Psi | H | \Psi \rangle \geq 0$

$\rightarrow$ can identify ground state $|0\rangle$ with

$$a(\vec{p})\,|0\rangle = 0$$
$$H\,|0\rangle = 0 \qquad\qquad \vec{P}\,|0\rangle = 0$$

excitations:

$$a^+(\vec{p})\,|0\rangle = |1_{\vec{p}}\rangle \quad \rightarrow E = E_p = \sqrt{\vec{p}^{\,2} + m^2}$$
$$\vec{P} = \vec{p}$$

$$a^+(\vec{p})\,a^+(\vec{q})\,|0\rangle = |1_{\vec{p}}, 1_{\vec{q}}\rangle \quad \rightarrow E = E_p + E_q$$
$$\vec{P} = \vec{p} + \vec{q}$$

$$a^+(\vec{p})\,a^+(\vec{q})\,a^+(\vec{q})\,|0\rangle = |1_{\vec{p}}, 2_{\vec{q}}\rangle$$

$\Rightarrow$ Excitations have properties of particles!

Fock space: Hilbert space spanned by all states $\{ |0\rangle, |1_{\vec{p}}\rangle, |1_{\vec{p}}, 1_{\vec{q}}\rangle \ldots \}$

comment:
$$\langle 1_{\vec p} | 1_{\vec p}\rangle = \langle 0| a(\vec p)\, a^\dagger(\vec p)\, |0\rangle$$
$$= \langle 0| a(\vec p)\, a^\dagger(\vec p) - a^\dagger(\vec p)\, \underbrace{a(\vec p)\, |0\rangle}_{=0}$$
$$= \langle 0| [a(\vec p), a^\dagger(\vec p)]\, |0\rangle$$
$$= (2\pi)^3\, 2E_p\, \underbrace{\delta^3(0)}_{\to\infty}\, \langle 0|0\rangle$$

→ states not normalizable,

can be fixed considering wave packets instead:

$$a_f^\dagger |0\rangle = \int d\vec k\; f(\vec k)\, a^\dagger(\vec k)\, |0\rangle$$

with square-integrable function f

→ norm $\langle 0| a_f\, a_f^\dagger |0\rangle = \int d\vec k\; |f(\vec k)|^2 < \infty$

Since $[a^\dagger(\vec p), a^\dagger(\vec q)] = 0$

$\Rightarrow$ $a^\dagger(\vec p)\, a^\dagger(\vec q)|0\rangle = a^\dagger(\vec q)\, a^\dagger(\vec p)\, |0\rangle$

$\Rightarrow$ states are symmetric under exchange of particles

$\Rightarrow$ Klein-Gordon particles are <u>bosons</u>

$\Rightarrow$ multiple particles can be in same state, e.g. $|2_{\vec p}\rangle$

Are there particles with negative energy?

The Fourier decomposition

$$\phi(x) = \int d\vec{p} \left[a(\vec{p}) e^{-ipx} + a^{+}(\vec{p}) e^{+ipx} \right]$$

contains plane-wave solutions with positive and negative energy:

$$e^{-iEt} \rightarrow E > 0$$

$$e^{+iEt} \rightarrow E < 0 \quad ?$$

Interpretation within QFT:

$a^{+}(\vec{p})$ creates particle with positive energy

$a(\vec{p})$ annihilates particle with pos. energy

$\Rightarrow$ $\phi(x)$ contains both creation and annihilation of the same particle type

$\Rightarrow$ real scalar fields are their own anti-particle examples for real scalar particles:

- Higgs boson ($\Rightarrow$ elementary)
- neutral pion π^0:

  bound state of u, d - quarks

  description of π^0 as scalar particle only works at low energies, where substructure not resolved.

Quantization of the complex scalar field

examples: charged pions $\pi^\pm$, kaons $K^0, \bar{K}^0$

$\rightarrow$ bound states of quarks

$\rightarrow$ description as scalar particles only at low energies

Fourier decomposition

$$\phi(x) = \int d\tilde{p} \left(a(\vec{p})\, e^{-ipx} + \underbrace{b^+(\vec{p})}_{\neq a^+,\ \text{since } \phi^+ \neq \phi} e^{+ipx} \right)$$

canonical commutator relations

$$[a(\vec{p}), a^+(\vec{p}\,')] = [b(\vec{p}), b^+(\vec{p}\,')] = (2\pi)^3\, 2E_p\, \delta^3(\vec{p}-\vec{p}\,')$$

all other commutators $= 0$

a^+	creates	particles	of type	a
b^+	"	"		b
a	annihilates	"		a
b	"	"		b

$\Rightarrow$ ϕ annihilates particle of type a,
creates particle of type b

ϕ^+ annihilates " b
creates " a

We can also associate states of type

$\begin{Bmatrix} a \\ b \end{Bmatrix}$ with $\begin{Bmatrix} \text{pos.} \\ \text{neg.} \end{Bmatrix}$ charges.

ϕ^+ is anti-particle of ϕ.

Quantization of the Dirac field

Solutions of Dirac equation:

- pos. energy solutions, spinors $u_s(\vec{p})$
- neg. " $v_s(\vec{p})$

with 2 spin states $s = \pm \frac{1}{2}$, each.

Fourier decomposition

$$\psi(x) = \int d\tilde{p} \sum_s \left(a_s(\vec{p})\, u_s(\vec{p})\, e^{-ipx} + b_s^\dagger v_s(\vec{p})\, e^{+ipx} \right)$$

$$\bar{\psi}(x) = \int d\tilde{p} \sum_s \left(a_s^\dagger(\vec{p})\, \bar{u}_s\, e^{ipx} + b_s\, \bar{v}_s(\vec{p})\, e^{-ipx} \right)$$

Imposing commutator relations for $a, a^\dagger, b, b^\dagger$ would lead to

$$H = \int d\tilde{p} \sum_s E_p \left(a_s^\dagger(\vec{p})\, a_s(\vec{p}) - b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right)$$

→ adding excitations of type b would decrease the energy. This could be done repeatedly!

→ no ground state

For the Dirac field, we instead impose <u>anti-commutator relations</u> $\{x,y\} = x\cdot y + y\cdot x$

$$\{ a_r(\vec{p}), a_s^{\dagger}(\vec{p}') \} = \delta_{rs} (2\pi)^3 \, 2E_p \, \delta^3(\vec{p}-\vec{p}')$$

$$= \{ b_r(\vec{p}), b_s^{\dagger}(\vec{p}') \}$$

$$\{ a, a \} = \{ a, b \} = \dots = 0$$

The energy and momentum is then given by

$$H = \int d\tilde{p} \sum_s E_p \Big(\underbrace{a^{\dagger}(\vec{p})\, a(\vec{p})}_{=:\, n_s^a(\vec{p})} + \underbrace{b^{\dagger}(\vec{p})\, b(\vec{p})}_{=:\, n_s^b(\vec{p})} \Big)$$

$$\vec{P} = \int d\tilde{p} \sum_s \vec{p} \left(n_s^a(\vec{p}) + n_s^b(\vec{p}) \right)$$

$\Rightarrow$ The Dirac field describes 2 types a, b of particles, e.g. electron and positron

$a_s^{\dagger}(\vec{p})$ creates electron e^- with $E_p > 0$, s, $\vec{p}$

$b_s^{\dagger}(\vec{p})$ " positron e^+ " $E_p > 0$, s, $\vec{p}$

1-particle states

$$|e^-(\vec{p},s)\rangle = a_s^+(\vec{p})\,|0\rangle$$

$$|e^+(\vec{p},s)\rangle = b_s^+(\vec{p})\,|0\rangle$$

Due to the anti-commutator relations we obtain

$$a_r^+(\vec{p})\,a_s^+(\vec{q})\,|0\rangle = -a_s^+(\vec{q})\,a_r^+(\vec{p})\,|0\rangle$$

$\Rightarrow$ states are antisymmetric

$\Rightarrow$ Dirac particles are fermions

In particular, we have

$$a_r^+(\vec{p})\,a_r^+(\vec{p})\,|0\rangle = 0$$

$\rightarrow$ Pauli exclusion principle

Quantization of the photon field

Fourier decomposition

$$A^\mu(x) = \int d\tilde{p} \sum_\lambda \left(\varepsilon^\mu_\lambda(\vec{p})\, a_\lambda(\vec{p})\, e^{-ipx} + \varepsilon^{*\mu}_\lambda(\vec{p})\, a^+_\lambda(\vec{p})\, e^{ipx} \right)$$

Only transverse polarizations are physical
$\rightarrow$ need to apply gauge fixing conditions

2 different methods

- impose radiation gauge on fields

$$A^0 = 0, \quad \vec{\nabla}\vec{A} = 0 \quad \rightarrow \quad \varepsilon^0_\lambda = 0, \quad \vec{p}\cdot\vec{\varepsilon}_\lambda(\vec{p}) = 0$$

$\Rightarrow$ only 2 non-zero polarizations $\varepsilon^\mu_1, \varepsilon^\mu_2$, but breaks Lorentz invariance

$$\left[a_\lambda(\vec{p}),\, a^+_{\lambda'}(\vec{p}\,')\right] = (2\pi)^3\, 2E_p\, \delta^3(\vec{p}-\vec{p}\,')\, \delta_{\lambda\lambda'}$$

- Covariant quantization (Gupta-Bleuler formalism) only impose gauge condition

$$\langle \text{phys} | \partial_\mu A^\mu | \psi'_{\text{phys}} \rangle = 0$$

on physical states

only contains transverse polarizations

but keep $\lambda = 0 \ldots 3$ in operator A^μ.

Quantization:

$$[a_\lambda(\vec{p}), a^\dagger_{\lambda'}(\vec{p}\,')] = \underbrace{-g_{\lambda\lambda'}} \, 2E_p \, \delta^3(\vec{p} - \vec{p}\,')$$

$$-g_{\lambda\lambda'} = \begin{cases} -\delta_{\lambda\lambda'} & \text{for } \lambda = 0 \\ \delta_{\lambda\lambda'} & \text{for } \lambda \neq 0 \end{cases}$$

$\rightarrow$ "wrong" sign for $\lambda = 0$

$\Rightarrow$ the contributions of ε_0^μ and ε_L^μ with $\vec{\varepsilon}_L \parallel \vec{p}$ cancel in physical quantities.