III.2. Perturbation theory & Feynman rules

So far we only considered the quadratic terms of the Lagrangian density

→ linear equations of motion,
exact solution given by Fourier decomposition

Higher terms in Lagrangian describe interactions

→ non-linear equations of motion

→ no exact solution,
use perturbation theory instead

The Dirac / Interaction picture

Hamiltonian

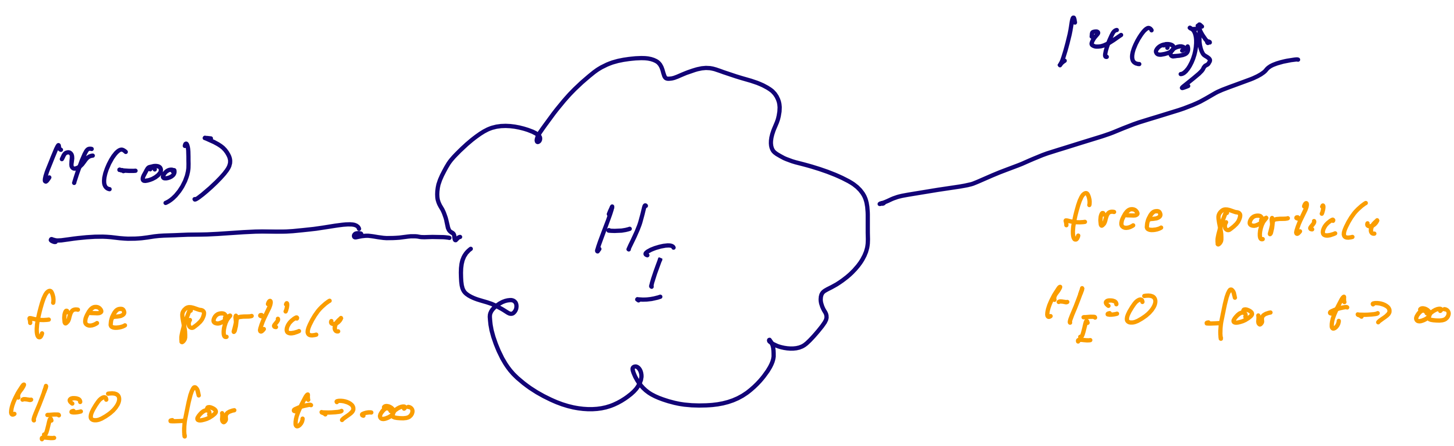
$$H = \underbrace{H_0}_{\text{free theory}} + \underbrace{H_I}_{\text{interactions}}$$

time dependence of states $|\psi\rangle$ and operators $\mathcal{O}$ given by

$$i\partial_t |\psi\rangle = H_I |\psi\rangle$$
$$i\partial_t \mathcal{O} = [\mathcal{O}, H_0]$$

$\rightarrow$ $\mathcal{O}(t)$ given by free theory $(H_I = 0)$
$|\psi(t)\rangle$ given by interaction H_I

Can be used to describe e.g. scattering experiments



$$T_1 < t < T_2$$

Interaction $H_I \neq 0$ only for limited time
$\rightarrow$ The states $|\psi(\pm\infty)\rangle$ are eigen states of H_0, with Fourier decomposition as discussed above

→ We can decompose

$$|i\rangle := |\psi(-\infty)\rangle = \sum_n a_n |n\rangle \quad , \sum |a_n|^2 = 1$$

$$|f\rangle := |\psi(\infty)\rangle = \sum_m b_m |m\rangle \quad , \sum |b_m|^2 = 1$$

into eigen states $|n\rangle$, $|m\rangle$ of H_0.

The scattering process is then described by an unitary transform S with

$$\boxed{|\psi(\infty)\rangle = \underline{S} \, |\psi(-\infty)\rangle}$$

<u>S - matrix</u>

Calculation of the S-matrix

The time dependence of the states is given by

$$i\partial_t |\psi(t)\rangle = H_I |\psi(t)\rangle$$

with initial condition $|\psi(-\infty)\rangle = |i\rangle$

formal solution:

$$|\psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 \; H_I(t_1)\,|\psi(t_1)\rangle$$

$$= |i\rangle + (-i)\int_{-\infty}^{t} dt_1 \; H_I(t_1)\,|i\rangle$$

$$+ (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \; H_I(t_1)\,H_I(t_2)\,|\psi(t_2)\rangle$$

$$= \dots$$

$$= |i\rangle + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n \; H_I(t_1)\dots H_I(t_n)\,|i\rangle$$

for $t \to \infty$, this contains the S-matrix

more compact notation for $t \to \infty$:

$$\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \; H_I(t_1)\,H_I(t_2)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \; T\left(H_I(t_1)\,H_I(t_2)\right)$$

with the time ordering operator T,

$$T\left(H_I(t_1)\,H_I(t_2)\right) = \begin{cases} H_I(t_1)\,H_I(t_2) & t_1 \geq t_2 \\ H_I(t_2)\,H_I(t_1) & t_2 > t_1 \end{cases}$$

$$\Rightarrow |\psi(\infty)\rangle = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \; T\left(H_I(t_1) \cdots H_I(t_n)\right) |i\rangle$$

$$= S \quad |i\rangle$$

$\Rightarrow$ S - matrix

$$S = T\, e^{-i\int_{-\infty}^{\infty} dt\; H_I(t)} = T\, e^{i\int_{-\infty}^{\infty} d^4x\; \mathcal{L}_I}$$

The probability to find a final state $|f\rangle$, given the initial state $|i\rangle$ is then given by

$$|\langle f | S | i \rangle|^2 \cdot \text{const.}$$

If the coupling constants are small, only few terms in the expansion of S are needed.

e.g. fine structure constant $\alpha = \frac{1}{137}$

Explicit calculation of S-matrix

We consider a real scalar field in ϕ^3-theory:

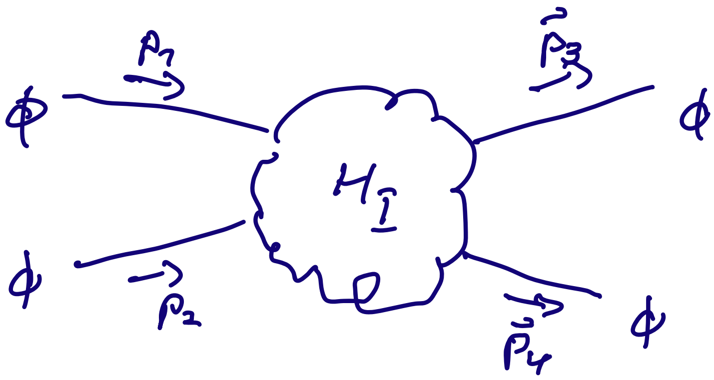
$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{3!} \phi^3}_{\mathcal{L}_I} \quad , \lambda \text{ small}$$

and consider the scattering of the initial state

$$|i\rangle = |1_{\vec{p}_1} 1_{\vec{p}_2}\rangle = a^+(\vec{p}_1)\, a^+(\vec{p}_2)\, |0\rangle$$

into the final state

$$|f\rangle = |1_{\vec{p}_3} 1_{\vec{p}_4}\rangle = a^+(\vec{p}_3)\, a^+(\vec{p}_4)\, |0\rangle$$



$$\Rightarrow S_{fi} = \langle f | S | i \rangle = \langle f | T\, e^{i \int d^4x \overbrace{\frac{-\lambda}{3!} \phi^3(x)}^{\mathcal{L}_I}} | i \rangle$$

first non-zero contributions at order λ^2:

$$S_{fi} = \langle f | \, T \, \frac{1}{2} \int_{-\infty}^{\infty} d^4x \, \frac{-i\lambda}{3!} \phi^3(x) \int_{-\infty}^{\infty} d^4y \, \frac{-i\lambda}{3!} \phi^3(y) \, | i \rangle$$

$$= \frac{1}{2} \left(\frac{-i\lambda}{3!} \right)^2 \int d^4x \int d^4y$$

$$\langle 0 | \, T \, a(\vec{p}_3) \, a(\vec{p}_4) \, \phi(x) \, \phi(x) \, \phi(x) \, \phi(y) \, \phi(y)$$

$$\underbrace{\left(\int d\tilde{k} \; a(\vec{k}) e^{-iky} + a^+(\vec{k}) e^{iky} \right)}_{\phi(y)} \, a^+(\vec{p}_1) \, a^+(\vec{p}_2) \, |0\rangle$$

$$\int d\tilde{k} \; a(\vec{k}) \, e^{-iky} \, a^+(\vec{p}_1) \, |0\rangle$$

$$= \int d\tilde{k} \; e^{-iky} \left(a^+(\vec{p}_1) \underbrace{a(\vec{k}) |0\rangle}_{=0} + [a(\vec{k}), a^+(\vec{p}_1)] \, |0\rangle \right)$$

$$= \int d\tilde{k} \; e^{-iky} \, (2\pi)^3 2E_k \, \delta^3(\vec{k} - \vec{p}_1) \, |0\rangle$$

$$d\tilde{k} = \frac{d^3\vec{k}}{(2\pi)^3 2E_k}$$

$$= e^{-ip_1 y} \, |0\rangle$$

$\rightarrow$ can be applied to 4 of the 6 factors

$(\phi(x))^3 \quad (\phi(y))^3$

$$S_{fi} = \frac{1}{2}\left(\frac{-i\lambda}{3!}\right)^2 \int d^4x \int d^4y \quad e^{-i(p_1+p_2)y} \, e^{+i(p_3+p_4)x}$$

$$\underbrace{\langle 0| T\, \phi(x)\, \phi(y) |0\rangle}_{=:\, D_F(x-y)} + \text{other contractions of operators } a, a^\dagger$$

The Feynman propagator

$$\boxed{\begin{aligned} D_F(x-y) &= \langle 0| T\, \phi(x)\phi(y) |0\rangle \\ &= \lim_{\varepsilon \to 0^+} \int \frac{d^4p}{(2\pi)^4} \underbrace{\frac{i}{p^2 - m^2 + i\varepsilon}}_{=:\, \tilde{D}_F(p)} e^{-ip(x-y)} \end{aligned}}$$

is the Green's function of the Klein-Gordon equation,

$$(\partial_\mu \partial^\mu + m^2) D_F(x) = -i\delta^4(x)$$

and describes the propagation of a particle from point x to point y.

The Feynman $i\varepsilon$ prescription guarantees the correct time ordering when integrating over the momentum p (and is only needed in loop diagrams $\rightarrow$ TTP 7-2)

In contrast to the free particle solutions, the momentum integral $\int d^4p$ is not restricted to the on-shell modes with

$$p^2 = E^2 - \vec{p}^{\,2} = m^2$$

$$S_{fi} = \frac{1}{2}\left(\frac{-i\lambda}{3!}\right)^2 \int d^4x \int d^4y \; e^{-i(p_1+p_2)y} \, e^{+i(p_3+p_4)x}$$

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip(x-y)} + \text{other contractions}$$

$$= \frac{1}{2}\left(\frac{-i\lambda}{3!}\right)^2 \int d^4x \int d^4y \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon}$$

$$e^{-i(p_1+p_2-p)x} \, e^{+i(p_3+p_4-p)y} + \text{other contr.}$$

$$\int dx \; e^{ikx} = 2\pi\,\delta(k)$$

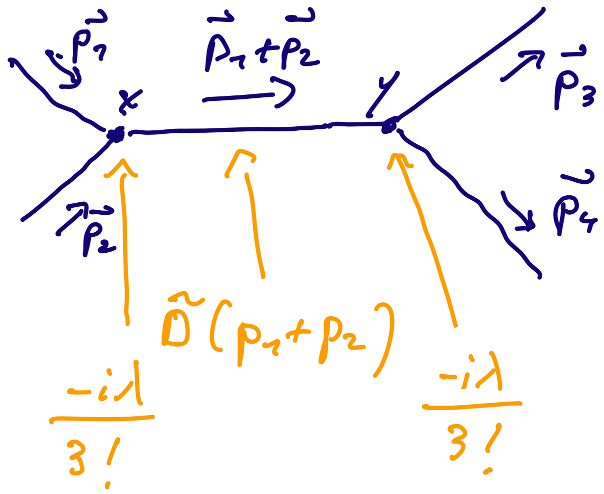
$$= \frac{1}{2}\left(\frac{-i\lambda}{3!}\right)^2 \int \frac{d^4p}{(2\pi)^4} (2\pi)^4 \delta^4(p_1+p_2-p)$$

$$\cdot (2\pi)^4 \, \delta^4(p_3 + p_4 - p) \cdot \frac{i}{p^2 - m^2 + i\epsilon} \quad + \text{ other contr.}$$

$$= \frac{1}{2} \left(\frac{-i\lambda}{3!}\right)^2 (2\pi)^4 \, \delta^4(p_1 + p_2 - p_3 - p_4) \cdot \frac{i}{(p_1+p_2)^2 - m^2 + i\epsilon}$$

$$+ \text{ other contr.}$$

$\rightarrow$ graphical representation



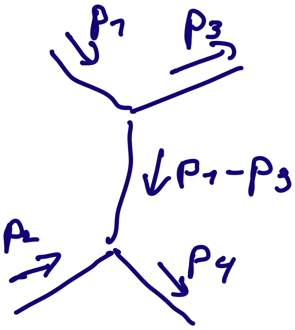
$+$ other contr.

other contractions:

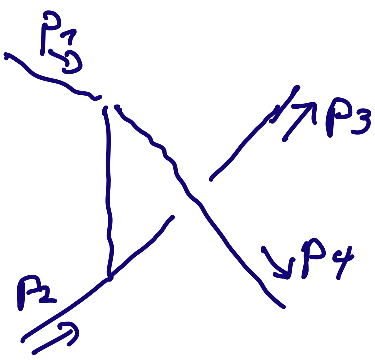
- $3!$ possibilities to contract $[\phi(x)]^3$ with $|1_{\vec{p}_1}, 1_{\vec{p}_2}\rangle$ and $\phi(y)$
  $\rightarrow$ cancels factor $\frac{1}{3!}$ from vertex, similarly for vertex y
- contractions with $y \leftrightarrow x$ swapped

give same result $\rightarrow$ cancel factor of $\frac{1}{2}$ from expansion of $e^{i\int \mathcal{L}}$

- contractions leading to different momentum configurations in vertices / propagators:

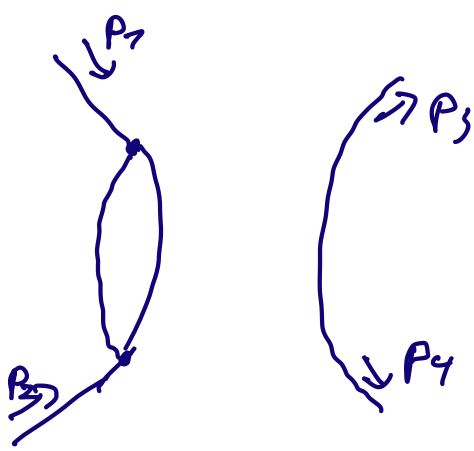


$$\sim \frac{i}{(p_1-p_3)^2 - m^2 + i\varepsilon}$$



$$\sim \frac{i}{(p_1-p_4)^2 - m^2 + i\varepsilon}$$

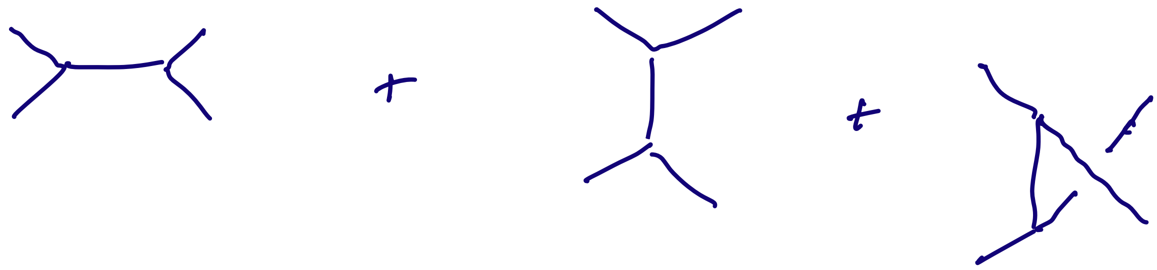
- disconnected graphs, e.g.



can be absorbed into definition of vacuum for interacting theory

$$\Rightarrow S_{fi} = (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \cdot (-i\lambda)^2$$

$$\cdot \left[\frac{i}{(p_1+p_2)^2 - m^2 + i\varepsilon} + \frac{i}{(p_1-p_3)^2 - m^2 + i\varepsilon} + \frac{i}{(p_1-p_4)^2 - m^2 + i\varepsilon} \right]$$



All factors can be obtained from graph representation of scattering process

$\Rightarrow$ Feynman rules