$\Rightarrow$ Feynman rules of ϕ^3 and ϕ^4 theory

$\mathcal{L}_I = -\frac{\lambda}{3!}\phi^3$ $\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4$

- Draw all connected graphs corresponding to the given initial and final state, with $\left\{ \begin{matrix} 3 \\ 4 \end{matrix} \right\}$ lines meeting at each vertex in $\left\{ \begin{matrix} \phi^3 \\ \phi^4 \end{matrix} \right\}$ theory.

- impose momentum conservation at each vertex;
  factor $(2\pi)^4 \delta^4(p_{in} - p_{out})$
  for overall momentum conservation.
  comment: Since the S-matrix always contains this δ-function, we usually only calculate the matrix element $\mathcal{M}$ with

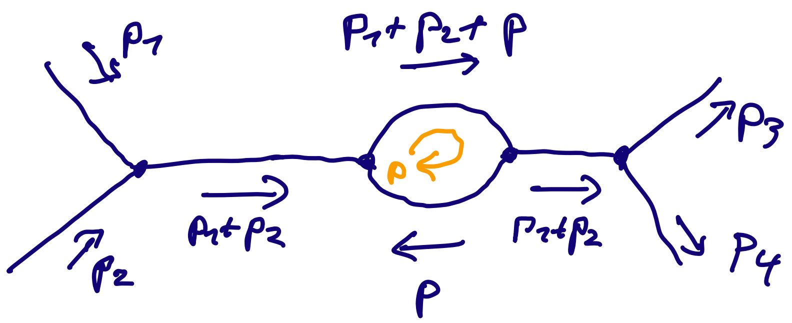
$$\langle f | S | i \rangle = (2\pi)^4 \delta^4(p_{in} - p_{out}) \langle f | i\mathcal{M} | i \rangle$$

- factor $-i\lambda$ for each vertex

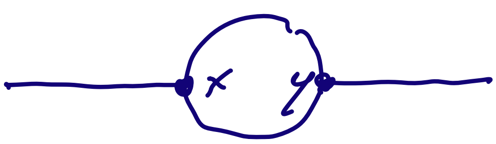
- factor $\tilde{D}(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$ for each internal line with momentum p

- $\int \frac{d^4 p}{(2\pi)^4}$ for each closed loop of the graph.

e.g. at order λ^4:



- additional symmetry factor, in case not all factors of $\frac{1}{3!}$ (or $\frac{1}{4!}$ in ϕ^4 theory) are cancelled by combinatorical factors, e.g.



$$\langle f | \, (\phi(x))^3 \, (\phi(y))^3 \, | i \rangle$$

$\Rightarrow$ 3 possibilities to contract $(\phi(x))^3$ with $\langle f|$

3 " $(\phi(y))^3$ with $|i\rangle$

only 2 possibilities to contract remaining fields:

$$\langle f| \, \phi(x) \, \phi(x) \, \phi(x) \, \phi(y) \, \phi(y) \, \phi(y) \, |i\rangle$$

$$\Rightarrow \left(\frac{1}{3!}\right)^2 \cdot 3 \cdot 3 \cdot 2 = \frac{1}{2} = \text{symmetry factor}$$

III 3. Quantum electrodynamics (QED)

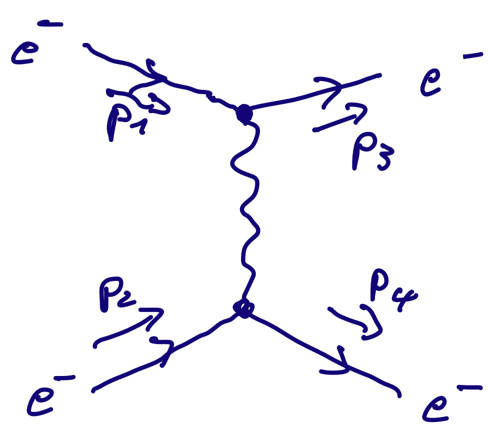
The interaction of Dirac fermions (e.g. electrons) with charge e with the electro-magnetic field is given by the QED Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e A_\mu \bar{\psi} \gamma^\mu \psi$$

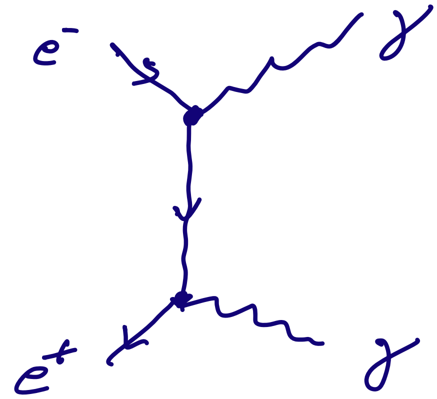
$\rightarrow$ sect. II.3

Similar to previous section, quantization and perturbative expansion leads to graphical representation of scattering process in Feynman graphs, e.g.

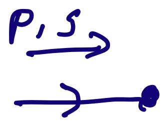
$$e^-(p_1) + e^-(p_2) \rightarrow e^-(p_3) + e^-(p_4)$$



$e^- + e^+ \rightarrow \gamma + \gamma$



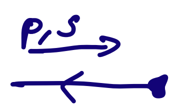
Feynman rules

- incoming electron e^-: 

$u_s(\vec{p})$

origin:

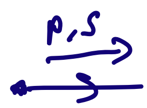
$\psi \,|e^-(p,s)\rangle \sim u_s(p)\, a_s(p)\; a_s^{\dagger}(p)\, |0\rangle$

  positron e^+: 

$\bar{v}_s(\vec{p})$

  photon γ: 

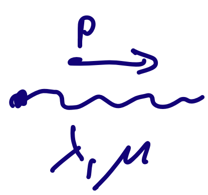
$\varepsilon_\mu(\vec{p}, \lambda)$

- outgoing electron e^-: 

$\bar{u}_s(\vec{p})$

  positron e^+: 

$v_s(\vec{p})$

  photon γ: 

$\varepsilon^*_\mu(\vec{p}, \lambda)$

comment: spinors $u, \bar{u}$ for e^-, $v, \bar{v}$ for e^+

$\bar{u}, \bar{v}$ for fermion flow out of diagram

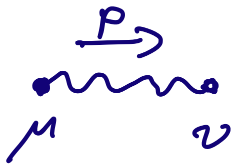
u, v " into "

- propagators

fermions (e^-, e^+)

$$\frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

photon γ



$$\frac{-i\, g_{\mu\nu}}{p^2 + i\varepsilon}$$

comments:

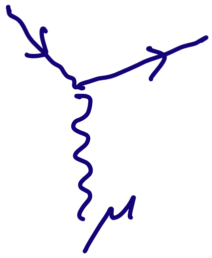
- for the fermion propagator, the direction of the momentum p needs to be the same as for fermion flow
- with $\not{p}\not{p} = p_\mu \gamma^\mu p_\nu \gamma^\nu$

$$= p_\mu p_\nu \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}$$

$$= p_\mu p_\nu \cdot g^{\mu\nu} = p^2$$

$$\rightarrow \frac{i}{\not{p} - m} = \frac{i}{\not{p} - m} \frac{\not{p} + m}{\not{p} + m} = \frac{i(\not{p} + m)}{p^2 - m^2}$$

$\frac{i}{\not{p} - m}$ should be understood as shortened notation for the latter expression.

- vertex  $= -ie\gamma^\mu$

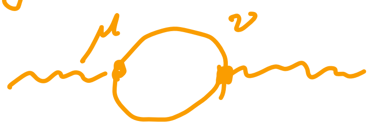
  origin: $\mathcal{L}_I = -e A_\mu \bar{\psi}\gamma^\mu \psi$

- momentum conservation at each vertex

- $\int \frac{d^4 l}{(2\pi)^4}$ for each closed loop

- factor (-1) for each closed fermion loop; apply trace to product of γ-matrices

  origin:

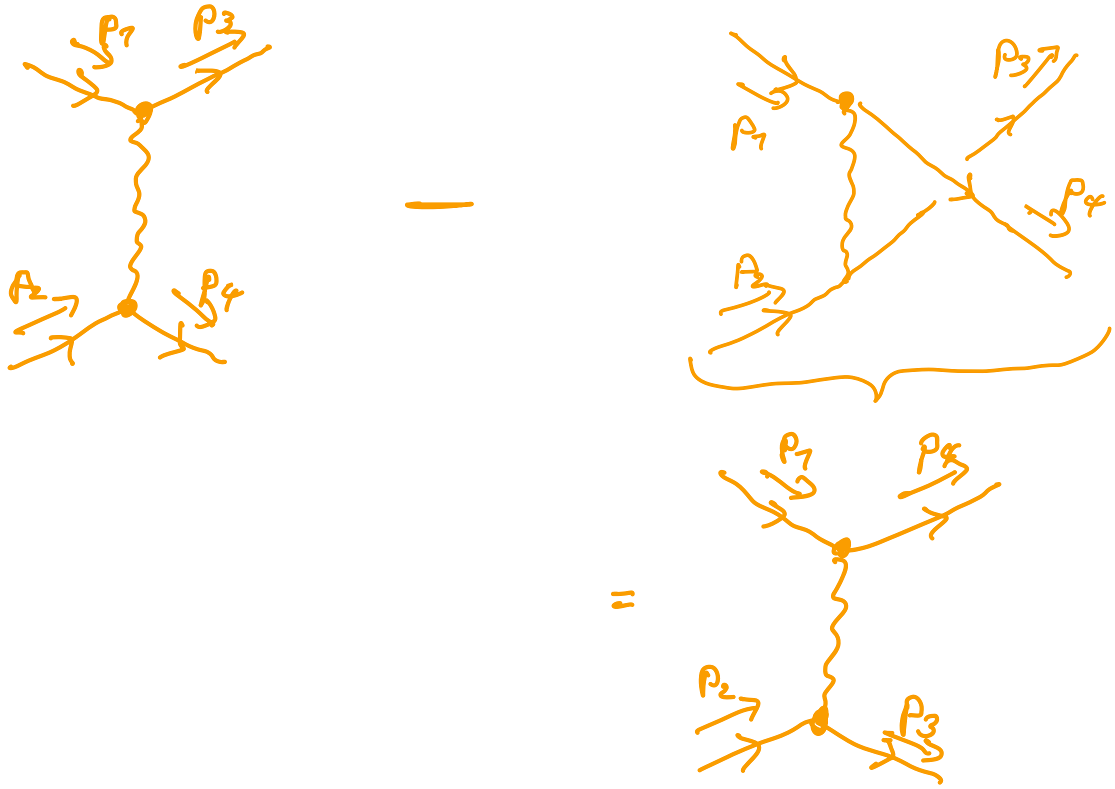
   $\sim \langle T\, \bar{\psi}\gamma^\mu\psi\, \bar{\psi}\gamma^\nu\psi \rangle$

  odd number of permutations required to rearrange this into product of propagators $\langle T\, \psi\bar{\psi} \rangle$

- relative minus sign for diagrams which differ only by interchange of external fermions

  $\rightarrow$ origin: antisymmetry of initial / final state

  e.g. $e^- e^- \rightarrow e^- e^-$



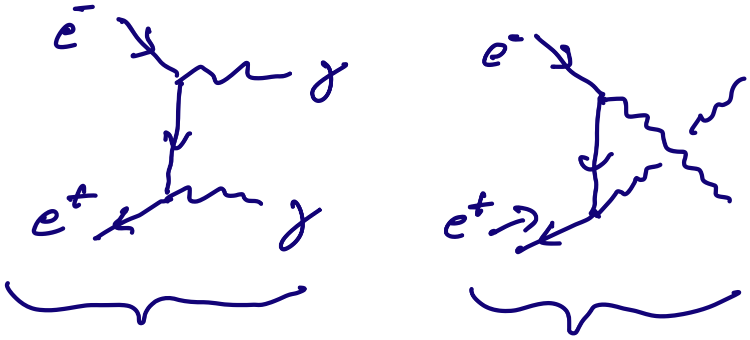
- along fermion lines, apply Feynman rules in direction opposite to fermion flow.

Applying these rules to each possible diagram for a given process gives the matrix element $i\mathcal{M}$. The S-matrix is then given by

$$\langle f | S | i \rangle = (2\pi)^4 \delta^4 (p_{in} - p_{out}) \langle f | i\mathcal{M} | i \rangle$$

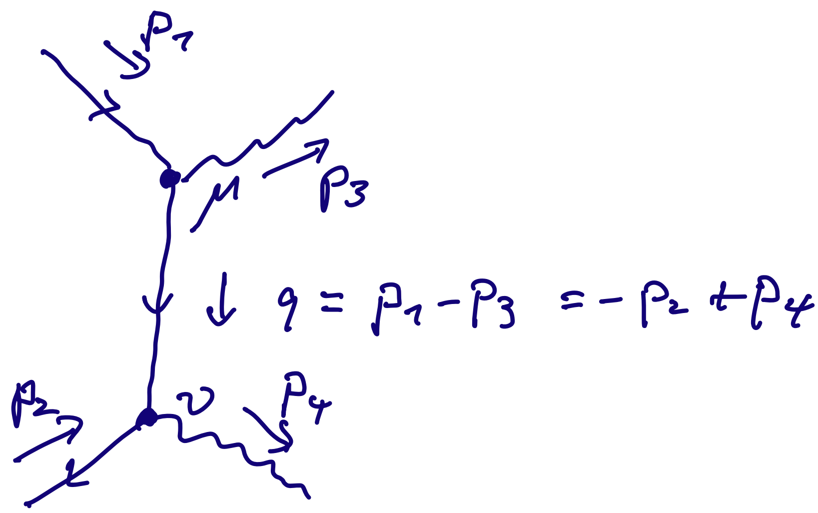
example : $e^- \, e^+ \rightarrow \gamma \, \gamma$

2 diagrams



$i\mathcal{M}_1$ $\quad$ $i\mathcal{M}_2$

diagram 1:



$$i\mathcal{M}_1 = \bar{v}(p,s_2)\,(-ie\gamma^\nu)\,\frac{i}{\not{q} - m + i\varepsilon}\,(-ie\gamma^\mu)\,u(p_1,s)$$

$$\cdot\, \varepsilon^*_\mu(p_3,\lambda_3)\;\varepsilon^*_\nu(p_4,\lambda_4)$$

$$= -ie^2\,\frac{\bar{v}_2\,\not{\varepsilon}_4\,(\not{q}+m)\,\not{\varepsilon}_3\,u_1}{q^2-m^2}$$

with $\bar{v}_2 = \bar{v}(p_2, s_2)$ u_1, $\varepsilon_3, \varepsilon_4 = \dots$