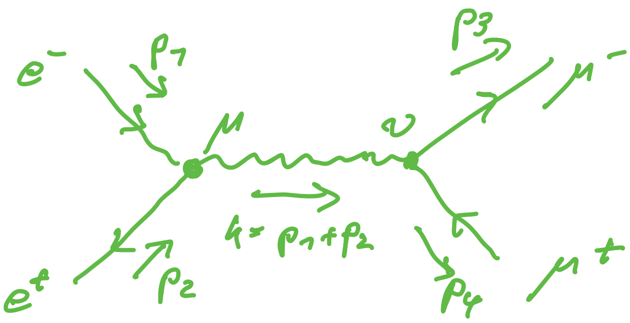
lecture 10:

differential cross section

$$d\sigma = \frac{1}{2w(s, m_1^2, m_2^2)} \cdot \prod_{i=1}^{n} d\tilde{q}_i \; (2\pi)^4 \delta^4\left(\sum_i q_i - p_1 - p_2\right) \cdot \left|\langle b_1 .. b_n | \mathcal{M} | a_1 a_2 \rangle\right|^2$$

$e^+e^- \to \mu^+\mu^-$



$$i\mathcal{M} = \bar{v}_2 \left(-ie\gamma^\mu\right) u_1 \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \bar{u}_3 \left(-ie\gamma^\nu\right) v_4$$

$$\Rightarrow \mathcal{M} = e^2 \; \bar{v}_2 \gamma^\mu u_1 \frac{g_{\mu\nu}}{k^2} \bar{u}_3 \gamma^\nu v_4$$

or with explicit spinor indices

$$\mathcal{M} = e^2 \; \bar{v}_{2\alpha} \gamma^\mu_{\alpha\beta} u_{1\beta} \frac{g_{\mu\nu}}{k^2} \bar{u}_{3\gamma} \gamma^\nu_{\gamma\delta} v_{4\delta}$$

To calculate $|\mathcal{M}|^2 = \mathcal{M}^+\mathcal{M}$, we need $\mathcal{M}^+$

$\Rightarrow$ need to calculate e.g.

$$\left(\bar{v}_2 \gamma^\mu u_1\right)^+ = \left(v_2^+ \gamma_0 \gamma^\mu u_1\right)^+$$

$$= u_1^+ \gamma^{\mu +} \gamma_0^+ v_2 = u_1^+ \gamma_0 \gamma^\mu \underbrace{\gamma_0 \gamma_0}_{\mathbb{1}} v_2$$

$$\left.\begin{aligned} \gamma^{0\dagger} &= \gamma^0 \\ \gamma^{i\dagger} &= -\gamma^i \end{aligned}\right\} \quad \gamma^{\mu\dagger} = \gamma_0 \gamma^\mu \gamma_0 \quad , \quad \gamma_0 \gamma_0 = \mathbb{1}$$

$$= \bar{u}_1 \gamma^\mu v_2$$

similarly : $\left(\bar{u}_3 \gamma^\nu v_4\right)^\dagger = \bar{v}_4 \gamma^\nu u_3$

$$\Rightarrow \quad \mathcal{M}^\dagger = e^2 \, \bar{u}_1 \gamma^\mu v_2 \, \frac{g_{\mu\nu}}{k^2} \, \bar{v}_4 \gamma^\nu u_3$$

$$= e^2 \, \bar{u}_{1\beta'} \gamma^{\mu'}_{\beta'\alpha'} v_{2\alpha'} \, \frac{g_{\mu'\nu'}}{k^2} \, \bar{v}_{4\delta'} \gamma_{\delta'\gamma'} u_{3\gamma'}$$

$$\Rightarrow |\mathcal{M}|^2 = \mathcal{M}^\dagger \mathcal{M}$$

$$= e^4 \, \bar{u}_1 \gamma^{\mu'} v_2 \, \frac{g_{\mu'\nu'}}{k^2} \, \bar{v}_4 \gamma^{\nu'} u_3$$

$$\bar{v}_2 \gamma^\mu u_1 \, \frac{g_{\mu\nu}}{k^2} \, \bar{u}_3 \gamma^\nu v_4$$

We need to calculate the spin sum / average of $|\mathcal{M}|^2$

$$\Sigma' |\mathcal{M}|^2 = \frac{1}{2 \cdot 2} \sum_{\substack{s_1, s_2, \\ s_3, s_4}} |\mathcal{M}|^2$$

$$= \frac{1}{4} \sum_{\text{spins}} e^4 \, \frac{g_{\mu'\nu'}}{k^2} \, \frac{g_{\mu\nu}}{k^2}$$

$$\cdot \, \bar{u}_{\beta'}(p_1, s_1) \, \gamma^{\mu'}_{\beta'\alpha'} \, v_{\alpha'}(p_2, s_2) \, \bar{v}_\alpha(p_2, s_2) \, \gamma^\mu_{\alpha\beta} \, u_\beta(p_1, s_1)$$

$$\cdot \, \bar{v}_{\delta'}(p_4, s_4) \, \gamma^{\nu'}_{\delta'\gamma'} \, u_{\gamma'}(p_3, s_3) \, \bar{u}_\gamma(p_3, s_3) \, \gamma^\nu_{\gamma\delta} \, v_\delta(p_4, s_4)$$

The sum over spins can be simplified with the relations

$$\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s) = (\not{p} + m\mathbb{1})_{\alpha\beta}$$

$$\sum_s v_\alpha(p,s)\,\bar{v}_\beta(p,s) = (\not{p} - m\mathbb{1})_{\alpha\beta}$$

similarly for photons:

$$\sum_\lambda \varepsilon^\mu(p,\lambda)\,\varepsilon^{*\nu}(p,\lambda) = -g^{\mu\nu}$$

+ terms that later on vanish in QED

$$\Rightarrow \sum{}'|\mathcal{M}|^2 = \frac{1}{4}e^4 \frac{g_{\mu'\nu'}}{k^2}\frac{g_{\mu\nu}}{k^2}$$

$$\cdot (\not{p}_1 + m_e)_{\beta\beta'}\,\gamma^{\mu'}_{\beta'\alpha'}\,(\not{p}_2 - m_e)_{\alpha'\alpha}\,\gamma^\mu_{\alpha\beta}$$

$$\cdot (\not{p}_4 - m_\mu)_{\delta\delta'}\,\gamma^{\nu'}_{\delta'\gamma'}\,(\not{p}_3 + m_\mu)_{\gamma'\gamma}\,\gamma^\nu_{\gamma\delta}$$

for matrices $M = A \cdot B$

$\to M_{ij} = A_{ik} B_{kj}$

$\to tr(M) = \sum_i M_{ii} = A_{ik} B_{ki}$

$$= \frac{1}{4}e^4 \frac{g_{\mu'\nu'}}{k^2}\frac{g_{\mu\nu}}{k^2}$$

$$\cdot\, tr\left[(\not{p}_1 + m_e)\,\gamma^{\mu'}\,(\not{p}_2 - m_e)\,\gamma^\mu\right]$$

$$\cdot\, tr\left[(\not{p}_4 - m_\mu)\,\gamma^{\nu'}\,(\not{p}_3 + m_\mu)\,\gamma^\nu\right]$$

Calculation of the traces with identities
($\to$ ex. sheet 7)

- $\mathrm{tr}(\mathbb{1}) = 4$
- $\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4\, g^{\mu\nu}$
- $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\rho\nu}\right)$
- $\mathrm{tr}(\gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_n}) = 0$ for n odd

$$\mathrm{tr}\left[(\underbrace{\not{p}_1}_{= p_{1\alpha}\gamma^\alpha} + m_e)\, \gamma^{\mu'} (\not{p}_2 - m_e)\, \gamma^\mu\right]$$

$$= \mathrm{tr}(\not{p}_1 \gamma^{\mu'} \not{p}_2 \gamma^\mu)$$
$$+ m_e\, \mathrm{tr}(\gamma^{\mu'} \not{p}_2 \gamma^\mu) - m_e\, \mathrm{tr}(\not{p}_1 \gamma^{\mu'} \gamma^\mu)$$
$$- m_e^2\, \mathrm{tr}(\gamma^{\mu'} \gamma^\mu)$$

$$= 4\left(p_1^{\mu'} p_2^\mu - p_1 p_2\, g^{\mu'\mu} + p_1^\mu p_2^{\mu'}\right)$$
$$- m_e^2\, 4\, g^{\mu'\mu}$$

similarly:

$$\mathrm{tr}\left[(\not{p}_4 - m_\mu)\, \gamma^{\nu'} (\not{p}_3 + m_\mu)\, \gamma^\nu\right]$$

$$= 4 \cdot \left(p_4^{\nu'} p_3^\nu - p_4 p_3\, g^{\nu'\nu} + p_4^\nu p_3^{\nu'} - m_\mu^2\, g^{\nu'\nu}\right.$$

$$\Rightarrow \sum{}' |\mathcal{M}|^2 = \frac{4e^4}{(k^2)^2} g_{\mu\nu} g_{\mu'\nu'}$$

$$\cdot \left(p_1^{\mu'} p_2^{\mu} - p_1 p_2 \, g^{\mu'\mu} + p_1^{\mu} p_2^{\mu'} - m_e^2 \, g^{\mu'\mu} \right)$$

$$\cdot \left(p_4^{\nu'} p_3^{\nu} - p_4 p_3 \, g^{\nu'\nu} + p_4^{\nu} p_3^{\nu'} - m_\mu^2 \, g^{\nu'\nu} \right)$$

$$= \frac{4e^4}{(k^2)^2}$$

$$\cdot \left(p_1^{\mu'} p_2^{\mu} - p_1 p_2 \, g^{\mu'\mu} + p_1^{\mu} p_2^{\mu'} - m_e^2 \, g^{\mu'\mu} \right)$$

$$\cdot \left(p_4^{\mu'} p_3^{\mu} - p_4 p_3 \, g^{\mu'\mu} + p_4^{\mu} p_3^{\mu'} - m_\mu^2 \, g^{\mu'\mu} \right)$$

scalar prod.   prod. of numbers

$$= \frac{4e^4}{(k^2)^2} \Big[(p_1 \cdot p_4) \cdot (p_2 p_3) \cdot 2 + (p_1 p_3)(p_2 p_4) \cdot 2$$

$$- 2 p_4 p_3 \cdot (m_e^2 + p_1 p_2) - 2 p_1 p_2 (m_\mu^2 + p_3 p_4)$$

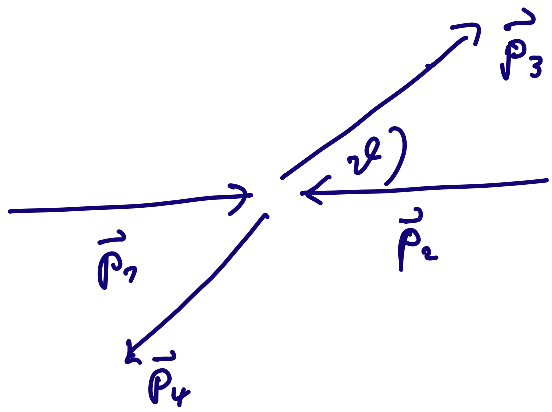
$$+ \underbrace{g^{\mu}_{\ \mu}}_{=4} (\underbrace{m_e^2}_{\approx 0} + p_1 p_2)(m_\mu^2 + p_3 p_4) \Big)$$

m_e negligible, since $\frac{m_e^2}{m_\mu^2} \approx \frac{1}{200^2}$

$$\approx \frac{8e^4}{(k^2)^2} \Big[(p_1 p_4)(p_2 p_3) + (p_1 p_3)(p_2 p_4) + m_\mu^2 \, p_1 p_2 \Big]$$

$\Rightarrow$ We have expressed $\sum' |\mathcal{M}|^2$ in terms of (Lorentz-invariant) scalar products of the external momenta.

We choose the center-of-mass system in the following



$|\vec{p}_1| = |\vec{p}_2|$

$|\vec{p}_3| = |\vec{p}_4|$

total energy:

$E_{CMS} = \sqrt{s}$

with $s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = k^2$

- incoming electrons, choose direction along z-axis

$$\vec{p}_1 = -\vec{p}_2 \quad , \qquad E_1 = E_2 = |\vec{p}_1| \qquad (\text{since } m_e \simeq 0)$$

$$\rightarrow \quad p_1 = \frac{\sqrt{s}}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad p_2 = \frac{\sqrt{s}}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

- outgoing muons:

$$E_{CMS} = \sqrt{s} = E_1 + E_2 = E_3 + E_4$$

with $E_3 = E_4 = \sqrt{|\vec{p}_3|^2 + m_\mu^2}$

$$\Rightarrow \quad |\vec{p}_3| = |\vec{p}_4| = \frac{\sqrt{s}}{2} \cdot \beta \quad , \quad \beta = \sqrt{1 - \frac{4m_\mu^2}{s}}$$

$$\Rightarrow \quad p_3 = \frac{\sqrt{s}}{2} \begin{pmatrix} 1 \\ \beta \cdot \sin\vartheta \cdot \cos\varphi \\ \beta \cdot \sin\vartheta \cdot \sin\varphi \\ \beta \cdot \cos\vartheta \end{pmatrix} \qquad p_4 = \frac{\sqrt{s}}{2} \begin{pmatrix} 1 \\ -\beta \sin\vartheta \cos\varphi \\ -\beta \sin\vartheta \sin\varphi \\ -\beta \cos\vartheta \end{pmatrix}$$

$\Rightarrow$ scalar products:

$$p_1 \cdot p_2 = \frac{s}{2}$$

$$p_1 \cdot p_3 = p_2 \cdot p_4 = \frac{s}{4} \cdot (1 - \beta \cdot \cos\vartheta)$$

$$p_2\, p_3 = p_1\, p_4 = \frac{s}{4} (1 + \beta \cdot \cos\vartheta)$$

$$\Rightarrow \sum{}' |\mathcal{M}|^2 = e^4 \cdot \left[1 + \beta^2 \cos^2\vartheta + (1 - \beta^2) \right]$$

We can now calculate the differential cross section:

$$d\sigma = \frac{1}{2w(s, m_1^2, m_2^2)} \, d\Phi_2 \cdot \sum{}' |\mathcal{M}|^2$$

- flux factor: $\dfrac{1}{2\, w(s,0,0)} = \dfrac{1}{2s}$

- phase space

$$d\Phi_2 = \frac{d^3\vec{p}_3}{(2\pi)^3\, 2E_3} \, \frac{d^3 p_4}{(2\pi)^3\, 2E_4} \, (2\pi)^4 \delta^4(k - p_3 - p_4)$$

$\downarrow$ exercise 6.2

$$= d\Omega \, \frac{1}{32\pi^2 k^2} \, \lambda(k^2, m_\mu^2, m_\mu^2)\, \Theta(k_0)\, \Theta(k^2 - 4m_\mu^2)$$

$$\lambda(a,b,c) = \sqrt{a^2 + b^2 + c^2 - 2ab - 2ac - 2bc}$$

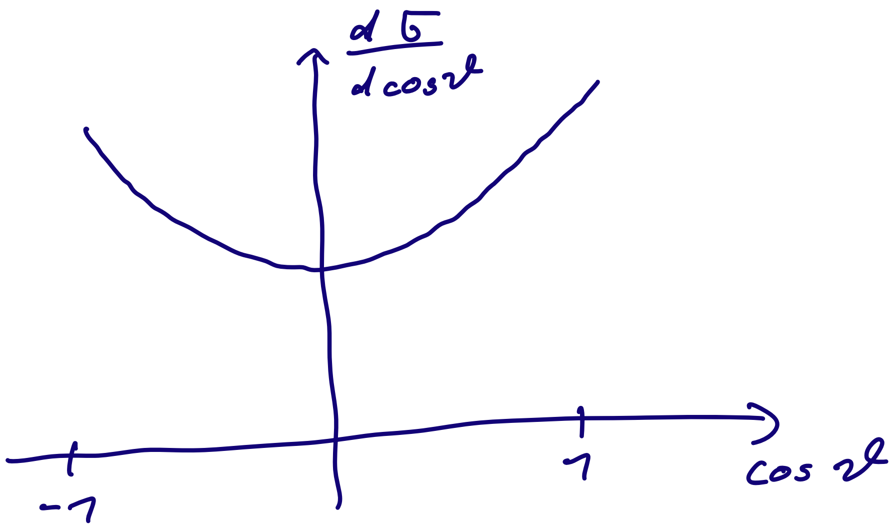
$$= d\Omega \quad \frac{1}{32\pi^2} \cdot \left(\beta \cdot \Theta(\sqrt{s}) \cdot \Theta(s - 4m_\mu^2)\right)$$

$d\cos\vartheta \; d\varphi$

process only possible, if $E_{CMS} = \sqrt{s} > 2m_\mu$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{2s} \cdot \frac{\beta}{32\pi^2} \, \Theta(\sqrt{s} - 2m_\mu)\, e^4 \left(1 + \beta^2 \cos^2\vartheta + (1-\beta^2)\right)$$

$\Rightarrow$ dependence on scattering angle ϑ



$\rightarrow$ angular dependence of final state

total cross section

$$\sigma = \int d\Omega \; \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \; \frac{\beta e^4}{64\pi^2 s} \left(1 + \beta^2\cos^2\vartheta + 1 - \beta^2\right)$$

consider $\sqrt{s} \gg m_\mu \quad \Rightarrow \quad \beta \approx 1$

fine structure constant $\quad \alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$

$$\Rightarrow \sigma(\sqrt{s} \gg m_\mu) = 2\pi \frac{\alpha^2}{4s}\left(2 + \frac{2}{3}\right) = \frac{4\pi\alpha^2}{3s}$$

JADE detector at PETRA collider,
DESY Hamburg, 1979-86

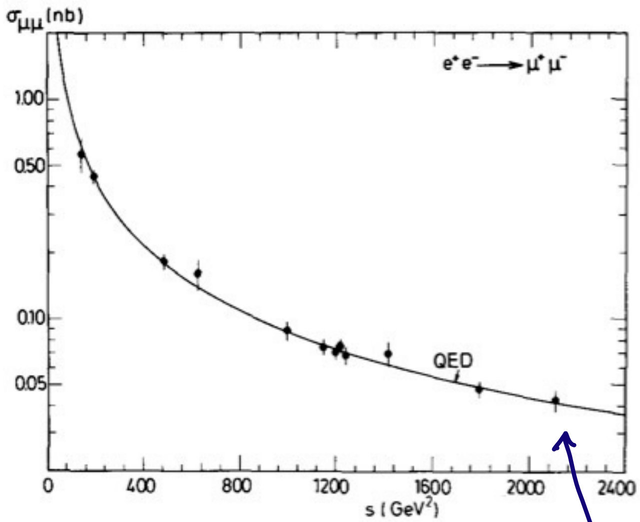


Fig. 1. Total cross section for $e^+e^- \to \mu^+\mu^-$ as a function of s, corrected for QED contributions up to order α^3

$\sqrt{s} \approx 46$ GeV

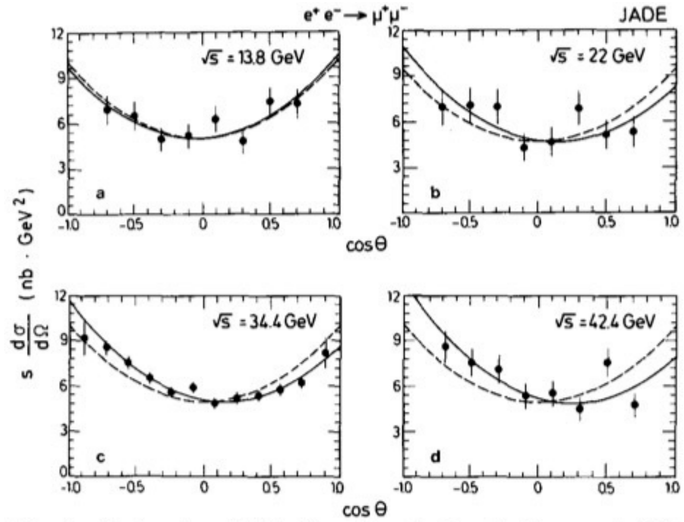
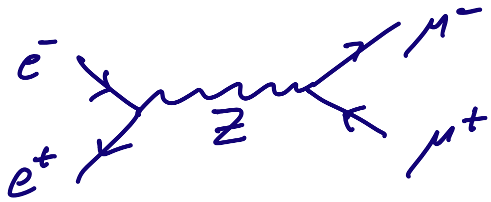


Fig. 2a–d. Angular distributions for $e^+e^- \to \mu^+\mu^-$ corrected for QED contributions to order α^3 for 4 cm energies. The full lines are fits allowing for an asymmetry, the dashed lines are symmetric fits

At high energies additional contribution of Z boson ($m_Z = 91$ GeV)



→ also leads to asymmetry in $\frac{d\sigma}{d\cos\vartheta}$

LEP, CERN arXiv: 0509008