Introduction to theoretical particle physics

I. 1. Introduction

Quantum Mechanics

- wave equations, describing systems with fixed number of particles
- relativistic equations lead to inconsitencies:
    - negative energy solutions
    - Klein paradox

  etc.

Quantum Field Theory (QFT)

- particles are identified as excitations of fields
  → creation / annihilation of particles possible
- QFTs we will consider:
    - scalar fields
    - Quantum - Electro - Dynamics (QED)
      → electromagnetic interactions

of a Dirac fermion (e.g. electron)

- Quantum-Chromo-Dynamics (QCD)
  → strong interactions of quarks and gluons, forming hadrons
- the standard model

• typical observables we will consider (using perturbation theory) are

- cross sections (total / partial)
- decay rates & branching fractions

The Standard Model (SM) of particle physics

• consistent description of all known elementary particles and their interactions, except for gravity.

• The interactions are direct consequences of local gauge symmetries of the theory:

$$\underset{\substack{\uparrow \\ \text{QCD}}}{SU(3)} \otimes \underbrace{SU(2) \otimes U(1)}_{\substack{\text{electromagnetism} \\ \text{\& weak interactions}}}$$

- particles are assigned to multiplets of these symmetry groups, leading to additional quantum numbers,

  e.g. el. charge, color, isospin

I.2. Natural Units

We set $\hbar = c = 1$

$\Rightarrow$ [energy] = [momentum] = [mass] = GeV

[length] = [time] = [energy]$^{-1}$ = GeV^{-1}

conversion to SI units by inserting appropriate factors of

$$c = 3 \cdot 10^8 \, \frac{m}{s}$$

$$\hbar = 6{,}6 \cdot 10^{-25} \, \text{GeV s}$$

$$\rightarrow \quad \hbar c \approx 200 \, \text{MeV} \cdot \text{fm}$$

$$= 0{,}2 \, \text{GeV} \cdot \text{fm}$$

$(\text{fm} = 10^{-15} \, m)$

cf.: $m_{\text{proton}} = 938 \, \text{MeV}$

$r_{\text{proton}} \approx 0.8 \, \text{fm}$

We also set $\varepsilon_0 = \mu_0 = 1$

vacuum permittivity (ε_0), magn. permeability (μ_0)

$\Rightarrow$ fine-structure constant

$$\alpha = \frac{e^2}{4\pi \varepsilon_0 \hbar c} = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

I.3. Covariant notation

contravariant 4-vectors: $a = a^\mu = (a^0, a^1, a^2, a^3) = (a^0; \vec{a})$

e.g. $x^\mu = (x^0, \vec{x}) = (ct, \vec{x})$

$p^\mu = (p^0, \vec{p}) = (E/c, \vec{p})$

covariant vectors: $a_\mu = (a_0, -\vec{a}) = g_{\mu\nu}\, a^\mu$

with metric $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$

$$\Rightarrow \quad g^\mu_{\ \nu} = g^{\mu\rho}\, g_{\rho\nu} = \mathrm{diag}\,(1,1,1,1) = \delta^\mu_\nu$$

$$g^\mu_{\ \mu} = 4$$

scalar product:

$$a \cdot b = a^0 b^0 - \vec{a}\cdot\vec{b} = a^\mu g_{\mu\nu} b^\nu = a_\mu b^\mu$$

invariant under Lorentz transformations

derivatives:

The derivative w.r.t. a contravariant vektor is a covariant vector:

$$\partial_\mu = \frac{d}{dx^\mu}$$

e.g.:

$$\partial_\mu (a\cdot x) = \partial_\mu (a_\nu \cdot x^\nu)$$

$$= a_\nu \left(\frac{d}{dx^\mu} x^\nu\right)$$

$$= a_\nu \cdot \delta^\nu_\mu = a_\mu$$

II Lagrange formalism & symmetries

II. 1. Euler - Lagrange equations

reminder: classical system of N particles with coordinates $q_i(t)$ described by

Lagrangian $$L = L(q_i, \underset{\uparrow}{\dot{q}_i}, t) = T - V$$
$$= \frac{dq_i}{dt}$$

The equations of motion follow from Hamilton's principle

$$\delta S = 0 \quad \text{with} \quad S = \int_{t_0}^{t_1} dt \; L(q_i, \dot{q}_i, t)$$

$\Rightarrow$ Euler - Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Generalization to fields

$$q_i \longrightarrow \phi(x^\mu)$$

Instead of N degrees of freedom (d.o.f.),

we have infinitely many d.o.f for the field ϕ, given by the field strength at each space-time point x^μ.
To obtain a Lorentz invariant formulation, we further need the replacement

$$\dot{q}_i \longrightarrow \partial_\mu \phi(x^\mu)$$

This allows us to write the action as

$$S = \int dt \; L = \int dt \int d^3\vec{x} \; \mathcal{L} = \int d^4x \; \mathcal{L}$$

with the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ in a Lorentz invariant form.

Equations of motion

We consider the variation of S under a variation of the field

$$\phi \to \phi + \delta\phi$$

[this also implies

$$\partial_\mu \phi \to \partial_\mu \phi + \partial_\mu(\delta\phi) \Rightarrow \delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$$]

with $\delta\phi = 0$ at boundary of integration.
(i.e. fixed boundary condition or requiring $\phi(x) \to 0$ for $x \to \infty$)

$$\delta S = S(\phi+\delta\phi\,,\ \partial_\mu(\phi+\delta\phi)) - S(\phi,\partial_\mu\phi)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\underbrace{\delta(\partial_\mu\phi)}_{=\partial_\mu(\delta\phi)} \right]$$

$\downarrow$ integration by parts

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi \right] + \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi \right]_{\partial V}}_{=0}$$

since $\delta\phi = 0$ at boundary ∂V

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi$$

The action is extremal if

$\delta S = 0$ for arbitrary $\delta\phi$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0}$$

Euler - Lagrange equation

(equation of motion for fields ϕ)

Generalization for multiple fields ϕ_i

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0$$