exercise classes probably on

Mon 15:45 – 17:15

Tue 9:45 – 11:15

Lecture 1:

Hamilton's principle $\delta S = 0$

$$\boxed{\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0 \qquad \text{Euler-Lagrange equation}}$$

Examples

- (real) scalar field $\phi(x)$

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$$

$$= \frac{1}{2} \underbrace{(\partial_\mu \phi)(\partial^\mu \phi)}_{g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi} - \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2} (\partial_t \phi)(\partial_t \phi) - \frac{1}{2} (\vec{\nabla} \phi)(\vec{\nabla} \phi) - \frac{1}{2} m^2 \phi^2$$

derivatives:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial}{\partial(\partial_\mu \phi)} \left(\frac{1}{2} \sum_{\alpha\beta} g^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi) \right)$$

$$= \frac{1}{2} \sum_{\alpha\beta} g^{\alpha\beta} \left(\delta^\mu_\alpha \, \partial_\beta \phi + \partial_\alpha \phi \, \delta^\mu_\beta \right)$$

$$= \partial^\mu \phi$$

$$\Rightarrow 0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

$$= -m^2 \phi - \partial_\mu \partial^\mu \phi$$

$$\Rightarrow \boxed{(\Box + m^2)\phi = 0 \quad \text{Klein-Gordon equation}}$$

- Dirac field $\psi(x)$

$$\mathcal{L} = i\bar{\psi}\,\not{\partial}\psi - m\bar{\psi}\psi$$

$$= i\bar{\psi}_\alpha \gamma^\nu_{\alpha\beta}(\partial_\nu \psi_\beta) - m\bar{\psi}_\alpha \psi_\alpha$$

with spinors $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_4 \end{pmatrix}$, $\bar{\psi} = \psi^+ \gamma^0$

2 spin states
pos. } energy solutions
neg.

and Dirac γ-matrices γ^μ

- $\gamma^\mu = (\gamma^\mu)_{\alpha\beta}$ $\quad \mu = 0,..3$, $\quad \alpha,\beta = 1...4$

  $\to$ 1 4x4 matrix for each μ

- Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$$

- with a 4-vector a^μ, we also write $a_\mu \gamma^\mu = \not{a}$

Apply Euler-Lagrange equations to each component ψ_i, $\bar{\psi}_i$:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_i} = i \gamma^{\nu}_{i\beta} \partial_\nu \psi_\beta - m \psi_i$$

$$\frac{\partial \mathcal{L}}{\partial \psi_i} = -m \bar{\psi}_i$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_i)} = 0 \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} = i \bar{\psi}_\alpha \gamma^{\mu}_{\alpha i}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_i)} \right) = i \gamma^{\nu}_{i\beta} \partial_\nu \psi_\beta - m \psi_i = 0$$

$$\Rightarrow \boxed{(i\not{\partial} - m)\psi = 0}$$

Dirac equation

similarly:

$$-m \bar{\psi}_i - \partial_\mu \left(i \bar{\psi}_\alpha \gamma^{\mu}_{\alpha i} \right) = 0$$

$$\Rightarrow \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

II.2 Symmetries

We consider (infinitesimal) transformations of the coordinates and fields

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu$$
$$\phi_i \to \phi'_i(\{\phi_i\}) = \phi_i + \delta\phi_i$$

These transformations are called symmetries if they leave the Lagrangian (more precisely: the action) invariant.

We classify them as

δx^μ : space-time symmetries
$\delta \phi_i$: internal symmetries

and call them local if δx^μ, $\delta\phi_i$ depend on x, otherwise they are called global.

Noether's theorem

For every continuous symmetry of the action S, there exists a conserved current of the field.

Space-time symmetries

We consider infinitesimal translations

$$x^\mu \to x^\mu + \delta x^\mu = x^\mu + \delta a^\mu$$

The resulting variation of the Lagrangian is:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \cdot \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \delta(\partial_\mu \phi)$$

with

- $\delta\phi = \frac{\partial \phi}{\partial x^\nu} \cdot \delta a^\nu = (\partial_\nu \phi)\, \delta a^\nu$

- $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi) = (\partial_\mu \partial_\nu \phi)\, \delta a^\nu$

- $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$

$$\Rightarrow \delta \mathcal{L} = \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) \cdot (\partial_\nu \phi)\, \delta a^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot (\partial_\mu \partial_\nu \phi)\, \delta a^\nu$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \right) \delta a^\nu$$

Instead of expressing $\delta\mathcal{L}$ with $\delta\phi$, we can also directly calculate:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu = \partial_\mu \mathcal{L}\, \delta a^\mu$$

$$= \partial_\mu \mathcal{L}\, \delta a^\nu \underbrace{g^\mu{}_\nu}_{=\delta^\mu_\nu}$$

$$\Rightarrow 0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) - g^\mu{}_\nu \mathcal{L} \right) \delta a^\nu$$

$\Rightarrow$ $\boxed{\begin{array}{l} \partial_\mu T^\mu{}_\nu = 0 \\ \text{with the energy-momentum tensor} \\ T^\mu{}_\nu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - g^\mu{}_\nu \mathcal{L} \end{array}}$

$\partial_\mu T^\mu{}_\nu = 0 \quad \Rightarrow$ conserved current for each index ν

$T^\mu{}_\nu$ contains the Hamiltonian density

$$\mathcal{H} = T^0{}_0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}(\partial_0 \phi) - \mathcal{L}$$

$\Rightarrow$ Hamiltonian $\quad H = \int d^3\vec{x}\ \mathcal{H} = \int d^3\vec{x}\ T^0{}_0$

cf. classical mechanics:

$$H(p, q) = p \cdot \frac{dq}{dt} - L$$

$\rightarrow$ conjugate momentum of field ϕ

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$$

The momentum of the field ϕ is given by

$$P_i = \int d^3\vec{x}\ T^0{}_i = \int d^3\vec{x}\ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}(\partial_i \phi)$$

$$= \int d^3\vec{x}\ \Pi\,(\partial_i \phi)$$

internal symmetries

We consider global transformations of the fields

$$\phi_i \to \phi_i + \delta\phi_i = \phi_i + \frac{\delta\phi_i}{\delta\omega} \cdot \delta\omega$$

one variable to parametrize transform

$$\delta\mathcal{L} = \sum_i \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)}_{=0 \text{ (E-L eq.)}} \delta\phi_i + \partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right)$$

If the transformation is a symmetry ($\delta\mathcal{L} = 0$) and all fields satisfy the E-L equations:

$\Rightarrow$ conserved current

$$\partial_\mu j^\mu = 0 \quad , \quad j^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \frac{\delta\phi_i}{\delta\omega}$$

Example

- global $U(1)$ transform of free Dirac field,

$$\psi \longrightarrow \psi' = e^{-i\alpha}\psi \qquad \longrightarrow \qquad \delta\psi = -i\,\delta\alpha\,\psi$$
$$\bar{\psi} \longrightarrow \bar{\psi}' = e^{+i\alpha}\bar{\psi} \qquad\qquad \delta\bar{\psi} = +i\,\delta\alpha\,\bar{\psi}$$

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi = i\bar{\psi}'\not{\partial}\psi' - m\bar{\psi}'\psi' = \mathcal{L}'$$

($\rightarrow$ we have a symmetry)

conserved current:

$$j^\mu = \sum_{i=1}^{4} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)} \cdot \frac{\delta\psi_i}{\delta\alpha} + \frac{\delta\bar{\psi}_i}{\delta\alpha}\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_i)}}_{=0}$$

$$= \sum_i \left(i\bar{\psi}\gamma^\mu\right)_i \cdot \left(-i\psi_i\right)$$

$$= \bar{\psi}\gamma^\mu\psi$$

$$0 = \partial_\mu j^\mu = \partial_t\rho + \vec{\nabla}\vec{j}$$

with $\rho = \bar{\psi}\gamma^0\psi = \psi^+ \underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}\psi$

$= \psi^+\psi = |\psi|^2$

$\rightarrow$ density of the field

$$\vec{j} = \bar{\psi}\vec{\gamma}\psi$$

$\rightarrow$ $\partial_\mu j^\mu = 0$ can be interpreted as continuity equation of the Dirac field

- local transform of Dirac field

$$\psi(x) \longrightarrow \psi'(x) = e^{-i\alpha(x)}\psi(x)$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = e^{+i\alpha(x)}\bar{\psi}(x)$$

With $\partial_\mu \psi' = (\partial_\mu e^{-i\alpha(x)})\psi(x)$

$+ e^{-i\alpha(x)}(\partial_\mu \psi(x))$

we get

$$\mathcal{L}' = \mathcal{L} \underbrace{- i \frac{\partial\alpha}{\partial x^\mu} i \bar{\psi}\gamma^\mu\psi}_{\delta\mathcal{L} = (\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi}$$

$\Rightarrow$ the local $U(1)$ transform is <u>no</u> symmetry of the free Dirac field.

$\Rightarrow$ If we demand a local $U(1)$ symmetry of the theory, we need to add additional terms / fields to the Lagrangian. This will lead to electrodynamics, where $\delta\mathcal{L}$ will be cancelled by gauge transformations of the photon field.