KIT Karlsruhe Institute of Technology

**SS 2022**

# Mathematical Methods of Theoretical Physics

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## Exercise Sheet 1

Issue: 27.04.2022 – Submission: 05.05.2022 – Discussion: 11.05.2022

*Please hand in at the end of the lecture on the 5th of May or online via ILIAS.*

### Exercise 1: Riccati equation (6 points)

In the first lecture we encountered the Riccati equation which is a non-linear first-order differential equation. In general it can be written as

$$y'(x) = a(x)y^2(x) + b(x)y(x) + c(x)\,, \tag{1.1}$$

where $a(x)$, $b(x)$ and $c(x)$ are arbitrary functions. We assume that $a(x) \neq 0$, as otherwise the Eq. (1.1) would be a linear first-order differential equation.

(a) Perform the substitution

$$y(x) = -\frac{w'(x)}{a(x)\,w(x)} \tag{1.2}$$

and show that the Riccati equation of Eq. (1.1) is equivalent to a linear second-order differential equation for the function $w(x)$, i.e.

$$w''(x) + p(x)w'(x) + q(x)w(x) = 0\,. \tag{1.3}$$

What are the functions $p(x)$ and $q(x)$?

Consider a special case of the Riccati equation with $c(x) = 0$. The resulting equation is often referred to as the Bernoulli equation.

(b) Perform the transformation defined in question (a) and show that the special case considered in here is equivalent to a first-order linear equation

$$u'(x) + p(x)u(x) = 0\,, \tag{1.4}$$

with $u(x) = w'(x)$. Write down a general solution for $w(x)$ in this case.

It was possible to write a general solution for function $w(x)$ in the previous question because we managed to *factor* the differential equation for $w(x)$, i.e.

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + p(x)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x}\right] w(x) = 0\,. \tag{1.5}$$

Once such a factoring of a differential equation is obtained, the general solution can be written in terms of iterated integrals.
Consider the differential equation of Eq. (1.3) and assume that it can be factored into

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} + g(x)\right] w(x) = 0\,. \tag{1.6}$$

(c) Find expressions for $f(x)$ and $g(x)$ – write your answer in terms of algebraic and differential equations, do not solve them.

## Exercise 2: Local analysis of differential equations (10 points)

Consider the differential equation

$$xy'' + (a + 1 - x)y' + ny = 0\,, \tag{2.1}$$

where $a \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. For $n \geq 0$, you may recognise this as the differential equation fulfilled by the generalised Laguerre polynomials that enter the solutions of the Schrödinger equation for the hydrogen atom.

(a) Find and classify the singular points of this differential equation.

(b) Construct the two solutions around the regular singular point(s) using the Frobenius method.

(c) [*Optional*] Apply the method of dominant balance to construct the leading behaviour of the solutions around the irregular singular point(s).