Mathematical Methods of Theoretical Physics

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Exercise Sheet 10

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Exercise 1: Boundary layer theory, 1st order non-linear DE (6 points)

The exercise aims at presenting an example of a first-order non-linear boundary layer problem.
Given the initial-value problem

$$(x - \epsilon\, y(x))\, y'(x) + x\, y(x) = e^{-x}\,, \qquad y(1) = 1/e\,, \qquad x \in [0,1]\,, \tag{1.1}$$

we want to determine a leading-order perturbative approximation to $y(0)$ as $\epsilon \to 0^+$, i.e. the value of the solution in this particular point.

(a) Plot the exact solution for different values of ϵ, showing that for $\epsilon \to 0$ the solution is almost flat except for a narrow region close to $x \approx 0$ [use for instance the command `NDSolve` in Mathematica].

We call the region where the solution changes rapidly *inner region* or *boundary layer*. Two standard approximations can be made in boundary layer theory:

- Outside the boundary layer, i.e. in the *outer region*, the derivatives of $y(x)$ that are multiplied by ϵ can be neglected, since the solution is slowly varying.
- Inside the boundary layer (inner region), the coefficient functions of the differential equation can be approximated by constants. This is justified by the fact that the boundary layer is extremely narrow.

(b) Solve the differential equation outside the boundary layer.

Near $x = 0$, the approximations done outside the boundary layer are no longer valid. However, since x is small, we are allowed to approximate e^{-x} by 1.

(c) Can we perform further approximations? Why? *Clue:* Consider that the function varies rapidly in the boundary layer.

(d) Solve the differential equation valid in the boundary layer considering x as the dependent variable. *Clue:* Treat y as the inverse function of x and thus $y' = 1/x'$. The result is an implicit equation for $y_{\text{in}}(x)$ of the form $x = f(y_{\text{in}})$.

The unknown constant appearing in the solution inside the boundary layer can be determined by asymptotically matching to the solution found outside the boundary layer.

(e) For $x = \mathcal{O}(\epsilon^{1/2})$, find the leading behaviour of the solution inside and outside the boundary layer as $\epsilon \to 0$, and then determine the unknown integration constant.

(f) Find a leading order implicit equation for $y_{\text{in}}(0)$. Compare the numerical values of $y_{\text{in}}(0)$ with the exact numerical solution in $x = 0$ for $\epsilon = 0.1$ and 0.01.

Exercise 2: Boundary layer theory, 2nd order linear DE (6 points)

Here, we want to apply boundary layer theory to some second-order, linear differential equations in the limit $\epsilon \to 0$. The examples in this exercise have a boundary layer around $x = 0$.

- Find an approximation for the differential equation in the outer region and determine the corresponding approximate solution.
- Find an approximation in the boundary layer. It may be helpful to introduce an "inner variable" as in the lecture.
- Fix the integration constants from the boundary conditions in the appropriate regions as well as through asymptotically matching the solutions in their overlap regions.

After performing your approximations you can use CAS to solve the resulting differential equations in each region.

(a) $0 = \epsilon y''(x) + \cosh(x)\, y'(x) - y(x)$, with $y(0) = y(1) = 1$ for $x \in [0, 1]$.

(b) $0 = \epsilon y''(x) + (x^2 + 1)\, y'(x) - x^3 y(x)$, with $y(0) = y(1) = 1$ for $x \in [0, 1]$.