Mathematical Methods of Theoretical Physics

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Exercise Sheet 11

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Exercise 1: WKB approximation (6 points)

Given the homogeneous second-order linear differential equation

$$\epsilon y''(x) + a(x)y'(x) + b(x)y(x) = 0\,, \tag{1.1}$$

where $a(x) > 0$ and with the boundary conditions $y(0) = A$ and $y(1) = B$, we would like to find an approximate solution in the limit $\epsilon \to 0^+$ on the interval $x \in [0, 1]$. Here we want to use the WKB method, i.e. making the ansatz

$$y(x) = \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right)\,. \tag{1.2}$$

(a) Insert Eq. (1.2) into the differential equation and use the method of dominant balance to find a valid relative scaling between the limits $\delta \to 0^+$ and $\epsilon \to 0^+$, i.e. $\delta \sim \epsilon^\alpha$.

(b) Solve the differential equation(s) for $S_n(x)$ order by order in δ up to $\mathcal{O}(\delta)$ ($n = 2$). Remember that there should be two linearly independent solutions for a second-order equation.

(c) Fix the integration constants using the boundary conditions.

(d) Specialise your results to $A = B = 1$ for the boundary conditions and to $a(x) = \cosh(x)$ and $b(x) = -1$ as well as $a(x) = 1 + x^2$ and $b(x) = -x^3$ for the coefficient functions. Compare your results (up to the appropriate order) to the results found in the question 2(a) on the previous exercise sheet.

Exercise 2: WKB analysis of a Sturm-Liouville problem (8 points)

Let us consider the following differential equation on the interval $x \in [0, \pi]$,

$$y''(x) + EQ(x)y(x) = 0 \qquad \text{with} \qquad Q(x) > 0\,, \tag{2.1}$$

and Dirichlet boundary conditions, $y(0) = y(\pi) = 0$. It has an infinite number of non-trivial solutions for discrete values of the parameter $E = E_1, E_2, E_3, \ldots$
The n-th eigenvalue, E_n, is associated with the eigenfunction $y_n(x)$. Our goal is to study the eigenvalues and eigenfunctions of Eq. (2.1).

(a) Show that the eigenvalues are non-degenerate, i.e. show that eigenfunctions having the same eigenvalue E are proportional to each other. *Hint:* Show that in case of Eq. (2.1), the Wronskian of this equation is a constant and, further, that it has to vanish once the boundary conditions are taken into account.

(b) Show that eigenfunctions for different eigenvalues are orthogonal to each other, i.e. show that

$$\langle y_n, y_m \rangle \equiv \int_0^\pi y_n(x) y_m(x) Q(x) \mathrm{d}x = 0\,, \tag{2.2}$$

where $Q(x)$ is the weight function present in Eq. (2.1). *Hint:* Use integration-by-parts twice.

(c) Consider $\varepsilon^2 = 1/E$ and derive the leading-order behaviour of the function $y(x)$. *Hint:* See similar problem: Exercise 1 of exercise sheet 2.

(d) Derive an expression for eigenvalues E_n based on the solution found in the previous question.

(e) Find the normalisation of the eigenfunctions of question (c), such that

$$\langle y_n, y_n \rangle = 1\,. \tag{2.3}$$

Consider a special case $Q(x) = (x + \pi)^4$.

(f) Give expressions for E_n and $y_n(x)$. Calculate numerical values for the first 50 eigenvalues.

(g) *Optional:* Use the Mathematica function `DEigenvalues` to calculate the first 50 eigenvalues of the system defined in Eq. (2.1). Compare these results with the ones obtained in the previous question.