Karlsruhe Institute of Technology

SS 2022

Mathematical Methods of Theoretical Physics

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Exercise Sheet 12

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Exercise 1: Boundary layer theory and multiple-scale analysis (6 points)

This exercise aims at presenting an example of a differential equation that can be solved both with boundary-layer method and with multiple-scale perturbation theory.
Let us consider the homogeneous second-order linear differential equation

$$\epsilon\, y''(x) + a\, y'(x) + b\, y(x) = 0\,, \tag{1.1}$$

where a and b are constants, $a > 0$, and with the boundary conditions $y(0) = A$ and $y(1) = B$. Our goal is to find an approximate solution in the limit $\epsilon \to 0^+$ on the interval $x \in [0, 1]$.

(a) Solve the differential equation exactly and plot it for different values of ϵ, a, b, A, B. Can you claim that this is a boundary-layer problem?

The problem exhibits two intrinsic scales: a short scale t, and a long scale $x = \epsilon\, t$, which describe the inner and the outer regions, respectively.

(b) To use multiple-scale analysis it is convenient to rewrite the differential equation in terms of the short scale. Perform the change of variable to pass from x to t.

(c) Assume a perturbative expansion for $y(t)$ of the form $y(t) = Y_0(t, x) + \epsilon\, Y_1(t, x) + \dots$, and find the corresponding differential equations for $Y_0(t, x)$ and $Y_1(t, x)$ (consider only the ϵ^0 and the ϵ^1 orders).

(d) Solve the system of differential equations derived in the previous question and identify those terms that give rise to secular contributions. Find appropriate conditions to eliminate such contributions. *Hint:* The secular term of the form $t\, e^{-t}$ is exponentially suppressed and does not need to be eliminated.

(e) Find the leading-order approximation for $y(t)$, imposing the boundary conditions. Compare the result with the exact solution.

The same problem can be tackled by exploiting boundary-layer theory.

(f) Find the differential equation valid in the outer and in the inner regions, explaining which approximation can be performed and solve them imposing the appropriate boundary condition.

(g) Match the inner and the outer solutions in the region $x = \mathcal{O}(\sqrt{\epsilon})$.

In this case it is possible to build a uniform approximate solution valid for all $0 \leqslant x \leqslant 1$. In particular we can introduce

$$y_{\text{unif}}(x) = y_{\text{out}}(x) + y_{\text{in}}(x) - y_{\text{match}}(x)\,, \tag{1.2}$$

with $y_{\text{match}}(x)$ corresponding to the asymptotic behaviour of the inner solution in the matching region.

(h) Compute the uniform approximation and compare it with the exact solution and the approximate solution found with the multiple-scale approach.

Exercise 2: WKB approximation and multiple-scale analysis (6 points)

In this exercise we consider an oscillator with a slowly varying frequency. The system is described by the following differential equation

$$y''(t) + \omega^2(\varepsilon t)\, y(t) = 0\,, \tag{2.1}$$

where ε is a small parameter. We want to find an approximate solution $y(t)$ and we will approach this problem in two ways: using the WKB method as well as the multiple-scale analysis.

(a) Introduce a variable $\tau = \varepsilon t$ and convert the differential equation in Eq. (2.1) into a problem that can be solved with the WKB method. Write the leading order solution – you can use relevant formulas from previous problem sheets.

(b) Introduce a new time variable $T = f(t)$ and show that Eq. (2.1) can be transformed into a form

$$\ddot{y} + y + \varepsilon(\text{some function of } y \text{ and } \dot{y}) = 0\,, \tag{2.2}$$

where dot denotes the new time derivative $\mathrm{d}/\mathrm{d}T$. What is the function $f(t)$? What is the form of the perturbation term (term multiplied by ε)? *Hint:* Find $f'(t)$ and express T as an integral.

(c) Use a multiple-scale ansatz treating the variables T and $\tau = \varepsilon t$ as independent,

$$y = Y_0(T, \tau) + \varepsilon Y_1(T, \tau) + \dots \tag{2.3}$$

and derive a sequence of partial differential equations

$$\frac{\partial^2 Y_0}{\partial T^2} + Y_0 = 0\,, \qquad \frac{\partial^2 Y_1}{\partial T^2} + Y_1 = -\frac{\omega'(\tau)}{\omega^2(\tau)}\frac{\partial Y_0}{\partial T} - \frac{2}{\omega}\frac{\partial^2 Y_0}{\partial T \partial \tau}\,. \tag{2.4}$$

Hint: Be careful about what is a total derivative and what is a partial derivative. Here $\omega'(\tau) \equiv \mathrm{d}\omega/\mathrm{d}\tau$.

The solution for the function Y_0 reads

$$Y_0 = A(\tau)e^{+iT} + A^*(\tau)e^{-iT}\,, \tag{2.5}$$

where $A(\tau)$ is responsible for the amplitude of oscillations.

(d) Require that the secular term in the partial differential equation for Y_1 vanishes. Show that it leads to a solution for $A(\tau)$ which makes Y_0 match the result obtained in question (a) using the WKB method.