SS 2022

# Mathematical Methods of Theoretical Physics

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## Exercise Sheet 2

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### Exercise 1: Approximate solutions (12 points)

Consider a function $y(x)$ that satisfies the following differential equation

$$y''(x) + x^2 y(x) = 0\,, \tag{1.1}$$

and the boundary conditions

$$y(10) = 0\,, \qquad\qquad y'(10) = 1\,. \tag{1.2}$$

Our goal is to construct approximate solutions to this equation for $x > 10$ and study their behaviour.

**Part 1 (3 points):** The point $x = 10$ is a regular point of this differential equation. We want to construct a solution for $y(x)$ around $x = 10$ based on this fact.

(a) Find a recursion relation for the Taylor series coefficients and give explicit values for the first two non-vanishing coefficients.

(b) Give the approximate location $x_*$ of the first root of the function $y(x)$ for $x > 10$.
Give the approximate value $y_{\max}$ of the first maximum of function $y(x)$ for $x > 10$.

**Part 2 (9 points):** Since $x = 10 \gg 1$, we can solve the differential equation in Eq. (1.1) by constructing a solution around $x = \infty$.

(c) Perform the transformation $x \to 1/\xi$ on Eq. (1.1) and write down the differential equation for the function $y(\xi)$. What type of point is $\xi = 0$ in this equation?

(d) Make an ansatz $y(\xi) = e^{iS(\xi)}$ and derive the differential equation for $S(\xi)$.

(e) Use the method of dominant balance to find an approximate solution for the function $S(\xi)$.
*Note:* You need to calculate the first two terms since the second term changes the asymptotic behaviour around $\xi = 0$, as was the case in the lecture.

(f) Rewrite the solution that you found in the previous question in terms of $x = 1/\xi$. Fix the integration constants using the boundary conditions in Eq. (1.2).

(g) Using the new solution for $y(x)$ repeat question (b) to find estimates for $x_*$ and $y_{\max}$.

(h) Derive the next correction to your solution $S(\xi)$, rewrite it as a function of $x = 1/\xi$ and fix boundary conditions using Eq. (1.2). How does this affect the position of the root, $x_*$? Use the new and old values of $x_*$ to get an idea of the error.

**Part 3 (optional, no points):** Computer algebra.

(i) Use `Mathematica` to plot the approximate solutions derived in Parts 1 and 2. How do the two solutions compare with each other?
*Hint:* You can use the recursion relation derived in question (a) to iteratively improve on the solution derived in Part 1 beyond what you derived there.

(j) Use `Mathematica` to solve the differential equation of Eq. (1.1).

- Plot the exact solution derived in `Mathematica` and the approximate solutions.
- Plot the difference between the exact solution and the solution derived in Part 2.

*Hint:* Make use of the function `DSolve`. The result will be written in terms of some special functions that `Mathematica` can evaluate numerically.

## Exercise 2: Asymptotic relations (3 points)

You have encountered the asymptotic relations $\sim$ and $\ll$ in the lecture. They are defined in the following way: given two functions $f(x)$ and $g(x)$ and a point $x_0$, we define $f(x) \ll g(x)$ in $x \to x_0$ ("$f(x)$ is much smaller than $g(x)$ in $x \to x_0$") if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \tag{2.1}$$

and that $f(x) \sim g(x)$ in $x \to x_0$ ("$f(x)$ is asymptotic to $g(x)$") if

$$f(x) - g(x) \ll g(x) \text{ in } x \to x_0\,. \tag{2.2}$$

Consider the functions

$$f_1(x) = \ln(x)\,, \qquad f_2(x) = \frac{x^3 - x^2 + 3x - 3}{x^2 + x - 2}\,, \tag{2.3}$$

$$f_3(x) = \exp\left(-x - \sqrt{x} + \frac{1}{2}\ln(x)\right)\,, \qquad f_4(x) = \cosh(x) - \sinh(x)\,, \tag{2.4}$$

$$f_5(x) = \exp(x^2)\,, \qquad f_6(x) = \exp(-x) + 1\,, \tag{2.5}$$

$$f_7(x) = 1\,, \qquad f_8(x) = \sqrt{1 + x}\,, \tag{2.6}$$

$$f_9(x) = x^2\,, \qquad f_{10}(x) = \exp(\exp(-x))\,, \tag{2.7}$$

$$f_{11}(x) = x^2 \exp(\sqrt{x})\,, \qquad f_{12}(x) = \exp(-x) + \frac{1}{x}\,, \tag{2.8}$$

and relate them using $\ll$ and $\sim$ at the point $x \to \infty$.