

Mathematical Methods of Theoretical Physics

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Exercise Sheet 3

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Exercise 1: Parabolic cylinder functions (7 points)

Consider the parabolic cylinder equation given by

$$y''(x) + \left(a + \frac{1}{2} - \frac{1}{4}x^2\right) y(x) = 0, \quad (1.1)$$

where a is a real parameter.

- (a) Derive the leading behaviour of $y(x)$ as $x \rightarrow +\infty$.
- (b) Find an asymptotic series that approximates the function $y(x)$ around $x = +\infty$.

In the following questions focus on the solution that vanishes at $x = +\infty$.

- (c) Consider the case where $a \in \mathbb{N}_+$. What can you say about the series derived in (b)?
- (d) Consider the case where $a \in \mathbb{R}$ and a is not a positive integer. What can you say about the series in this case? Perform the ratio test to determine its radius of convergence.

Exercise 2: Leading behaviour (6 points)

Find the leading behaviour as $x \rightarrow 0$ for the differential equations

- (a) $x^4 y'''(x) = y(x)$,
- (b) $x^6 y''(x) = e^x y(x)$.

Exercise 3: Exponentiation of asymptotic relations (2 points)

- (a) Give an example of an asymptotic relation $f(x) \sim g(x)$ for $x \rightarrow \infty$ that cannot be exponentiated, i.e., for which $\exp(f(x)) \sim \exp(g(x))$ as $x \rightarrow \infty$ is false.
- (b) Show that if $f(x) - g(x) \ll 1$ as $x \rightarrow \infty$, then $\exp(f(x)) \sim \exp(g(x))$ as $x \rightarrow \infty$.