Mathematical Methods of Theoretical Physics

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Exercise Sheet 4

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Exercise 1: Stieltjes integrals and series (7 points)

In this exercise, we will consider a general case of the Stieltjes integral. To this end let us define the function $f(x)$ as

$$f(x) = \int_0^\infty \frac{\rho(t)}{1+xt}\mathrm{d}t\,, \tag{1.1}$$

where the weight function $\rho(t)$ is non-negative and such that all moments A_n, defined as

$$A_n = \int_0^\infty t^n \rho(t)\mathrm{d}t\,, \tag{1.2}$$

exist for all positive integers n.

(a) Show that the function $f(x)$ has the asymptotic expansion

$$f(x) \sim \sum_{n=0}^{\infty}(-1)^n A_n x^n\,, \tag{1.3}$$

as $x \to 0$.
Hint: Verify that the error $\epsilon_N(x)$, defined in the lectures, satisfies the asymptotic relation $\epsilon_N(x) \ll x^N$ for each positive integer N.

Consider the specific case with $\rho(t) = K_0(t)$, where $K_0(t)$ is the modified Bessel function that can be expressed as

$$K_0(x) = \int_1^\infty e^{-xt}(t^2-1)^{-1/2}\mathrm{d}t\,. \tag{1.4}$$

(b) Show that in this case

$$f(x) \sim \frac{1}{2}\sum_{n=0}^{\infty}(-2x)^n\,\Gamma^2\left(\tfrac{1}{2}n+\tfrac{1}{2}\right)\,. \tag{1.5}$$

Hint: Make use of the identity $\int_1^\infty (t^2-1)^{-1/2}t^{-n-1}\mathrm{d}t = 2^{n-1}\Gamma^2(n/2+1/2)/n!$.

(c) Along similar lines as in the first subproblem show that

$$\int_0^\infty \frac{\mathrm{e}^{-t}}{1+xt^2}\,\mathrm{d}t \sim \sum_{n=0}^{\infty}(2n)!\,(-x)^n \tag{1.6}$$

as $x \to 0$.

Exercise 2: The error function (7 points)

The error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t\, \mathrm{e}^{-t^2} , \tag{2.1}$$

which cannot be expressed in terms of elementary functions. The goal here will be to derive series expansions for this function.

(a) Derive a series expansion for $\operatorname{erf}(x)$ around $x = 0$ by expanding the integrand and integrating term-wise. Show that this series has an infinite radius of convergence.

(b) *(optional)* The series derived in the previous subproblem converges very slowly for large x. Use a computer algebra system to calculate the truncation of this series for different numbers of terms and different values of x. Try $x \in \{0.5, 1, 2, 5\}$ and determine after how many terms the difference between the truncated series and the numerical evaluation of $\operatorname{erf}(x)$ drops below 10^{-20}.

Instead of this convergent power series, we can also derive an asymptotic expansion around $x \to \infty$.

(c) Show that $\lim_{x\to\infty} \operatorname{erf}(x) = 1$.

(d) Using the result of the previous subproblem, we can write

$$1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{d}t\, \mathrm{e}^{-t^2} . \tag{2.2}$$

Use integration-by-parts to derive an asymptotic series in $1/x$. *Hint:* $\int \mathrm{e}^{-t^2} \mathrm{d}t$ cannot be expressed in terms of elementary functions, but $\int t\mathrm{e}^{-t^2} \mathrm{d}t$ can be.

(e) Show that the asymptotic series you derived in the previous subproblem is indeed asymptotic to $1 - \operatorname{erf}(x)$ as $x \to \infty$.

(f) Show that the radius of convergence for the asymptotic series is zero.

(g) *(optional)* Use a computer algebra system to calculate numerical values for different truncations of the asymptotic series. Look again at $x \in \{0.5, 1, 2, 5\}$ and determine after how many terms the difference to $\operatorname{erf}(x)$ is the smallest.