Mathematical Methods of Theoretical Physics

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Exercise Sheet 5

Issue: 25.5.2022 – Submission: 1.6.2022 – Discussion: 15.6.2022

Exercise 1: Integrand expansion (4 points)

Calculate the following integrals by expanding their integrands and then integrating term by term. Sum the resulting series if possible.

(a) $I_1 = \int_0^1 \frac{\mathrm{d}x}{x} \ln\left(\frac{1+x}{1-x}\right)$ (*Hint:* expand around $x \to 0$)

(b) $I_2 = \int_0^\infty \frac{\mathrm{d}x}{\cosh x}$ (*Hint:* rewrite and expand around $x \to \infty$)

For the next two integrals, find the first two terms of the leading behaviour in the limit $x \to 0$ (with $x > 0$).

(c) $I_3(x) = \int_x^1 \mathrm{d}t\, \cos(xt)$

(d) $I_4(x) = \int_0^1 \mathrm{d}t\, \sqrt{\sinh(xt)}$

Exercise 2: The Fresnel integrals (6 points)

In this exercise we will consider the Fresnel integrals defined as

$$C(x) = \int_x^\infty \cos(t^2)\,\mathrm{d}t\,, \qquad S(x) = \int_x^\infty \sin(t^2)\,\mathrm{d}t\,, \tag{2.1}$$

for $x > 0$. Our goal is to study these integrals in various limits.

(a) Show that

$$S(0) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,, \qquad C(0) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,. \tag{2.2}$$

Hint: Calculate the integral $\int_0^\infty e^{it^2}\mathrm{d}t$. Use Cauchy's theorem with a particular contour on the complex plane and the Gaussian integral.

(b) Find the full expansions for $S(x)$ and $C(x)$ as $x \to 0$. After how many terms for $x = 10$ does the next term in the series drop below 10^{-10}?
Hint: Use the result of question (a) and study the integrals over $[0, x]$.

(c) Find the asymptotic expansion for $S(x)$ and $C(x)$ as $x \to \infty$ using the integration-by-parts technique. After how many terms for $x = 10$ does the next term in the series drop below 10^{-10}?

Exercise 3: The Laplace method (5 points)

The goal of this exercise is to use Laplace's method to find asymptotic expressions for given integrals.
Consider the following integral

$$I(x) = \int_0^{10} (1+t)^{-1} e^{-xt} \mathrm{d}t \,. \tag{3.1}$$

(a) Find the leading behaviour of $I(x)$ for $x \to \infty$. Why does the result not depend on the parameter ϵ (see section 5 of the lecture notes)?

(b) Find the full asymptotic expansion of $I(x)$ for $x \to \infty$. Explain why it is possible to replace the parameter ϵ by ∞.

(c) Determine the leading behaviour of the following integrals as $x \to \infty$

$$I_1(x) = \int_{-\pi/2}^{\pi/2} (t+2)\, e^{-x\cos t} \mathrm{d}t \,, \tag{3.2}$$

$$I_2(x) = \int_0^{\infty} \frac{e^{-x\cosh t}}{\sqrt{\sinh t}} \mathrm{d}t \,. \tag{3.3}$$