Mathematical Methods of Theoretical Physics

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Exercise Sheet 6

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Exercise 1: Method of stationary phase (4 points)

Find the leading behaviour for $x \to \infty$ of the integrals given below by applying the method of stationary phase. Recall that the key steps of this method are:

- Identify the stationary point and isolate its contribution by modifying the integration domain by means of a small parameter ϵ.
- Expand the function appearing in the exponential up to the first non-trivial derivative around the stationary point and approximate the function in front of the exponential.
- Find a convenient change of variables to linearise the function appearing in the exponential.
- To simplify the calculation of the approximated integral, extend the integration domain to infinity.
- Apply the Cauchy theorem.

(a) $I_1(x) = \int_0^1 \mathrm{e}^{ixt^2} \cosh(t^2)\,\mathrm{d}t,$

(b) $I_2(x) = \int_0^1 \mathrm{e}^{ixt^4} \tan t\,\mathrm{d}t,$

(c) $J_n(x) = \int_0^1 \cos(n\pi t - x\sin(\pi t))\,\mathrm{d}t$ for $n \in \mathbb{N}_0,$

Exercise 2: Method of steepest descent (13 points)

Consider the integral

$$I(x) = \int_0^1 \mathrm{e}^{-4xt^2} \cos(5xt - xt^3)\mathrm{d}t\,. \tag{2.1}$$

Our goal is to find the leading behaviour of the integral $I(x)$ as $x \to +\infty$. We will find that the final result is given by

$$I(x) = \frac{\mathrm{e}^{-2x}}{2}\sqrt{\frac{\pi}{x}}\,. \tag{2.2}$$

We will approach this task in a few different ways.

(a) We first try a naive approximation. Since the exponent heavily dampens the result, we could be lead to expect that the leading behaviour is dominated by the region close to the lower integration boundary. Based on this assumption and inspired by the Laplace method, try to calculate the behaviour of the integral by inserting $t = 0$ everywhere except in the exponent. Compare to Eq. (2.2). What is the flaw in this approach?

For further analysis it is better to write

$$I(x) = \frac{1}{2}\int_0^1 \mathrm{e}^{-4xt^2}\left(\mathrm{e}^{+i5xt-ixt^3} + \mathrm{e}^{-i5xt+ixt^3}\right)\mathrm{d}t$$
$$= \frac{1}{2}\int_{-1}^1 \mathrm{e}^{-4xt^2+i5xt-ixt^3}\mathrm{d}t\,. \tag{2.3}$$

(b) Neglect the ixt^3 term and find the leading behaviour using the steps from the Laplace method. Again, compare to Eq. (2.2). Is this approach justified? Explain.

Let us now instead apply the method of steepest descent. Here, our first task is to replace the integration contour by a better suited contour with stationary phase by applying the Cauchy theorem.

(c) Rewrite the integral from Eq. (2.3) in the form of

$$I(x) = \frac{1}{2}\mathrm{e}^{-2x}\int \mathrm{e}^{x\rho(t)}\mathrm{d}t\,. \tag{2.4}$$

What is $\rho(t)$? Set $t = u + iv$ and write $\rho = \phi + i\psi$. Find two contours C_1 and C_2 of stationary phase that connect to the end points at $t = -1$ and $t = +1$. Why is it not possible to find a single stationary-phase contour that connects the two end points?

(d) The two contours derived in the previous subproblem run from ∞ through $t = \pm 1$ to ∞. Draw the contours. Which parts of the contours are useful for the steepest-descent method?

Remember that our goal is to apply the Cauchy theorem and to replace our initial contour by a sequence of steepest-descent/stationary-phase contours. The two stationary-phase contours found previously can be connected by a third contour C_3 that runs from ∞ to ∞. It is useful to choose the contour C_3 such that it runs through a saddle point.

(e) Find the saddle point(s) of the integrand of $I(x)$ and draw a contour of stationary phase that runs though one of them (which one?).

(f) Evaluate the contribution of the saddle point: find an approximation for both the contour and the integrand around the saddle point and simplify the calculation by extending the integration to infinity again.

(g) Find the leading behaviour of $I(x)$ as $x \to +\infty$ assuming that it is determined by the contribution of the saddle point. Is it justified to neglect the contributions of the integrals along the contours C_1 and C_2?